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DESIGN INFORMATION TECHNOLOGY OF QUANTUM ALGORITHM GATES

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IT design of quantum algorithmic gates (QAG) is considered. General structures of the QAG design method and simulation system are introduced. Applications to efficient simulation of quantum algorithms (QA) on classical computer are described.

Keywords: IT design of quantum algorithmic gates, efficient simulation of quantum algorithms, general structure of the quantum algorithms.

ИНФОРМАЦИОННАЯ ТЕХНОЛОГИЯ ПРОЕКТИРОВАНИЯ КВАНТОВЫХ АЛГОРИТМИЧЕСКИХ ЯЧЕЕК

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Рассмотрена информационная технология проектирования квантовых алгоритмических ячеек. Описываются общие структуры методов проектирования квантовых алгоритмических ячеек и их моделирования. Приведены приложения эффективного моделирования квантовых алгоритмов на классическом компьютере.

Ключевые слова: информационная технология проектирования квантовых алгоритмических ячеек, эффективное моделирование квантовых алгоритмов, общая структура квантовых алгоритмов.

General structure of the quantum algorithmic gate (QAG) design method

Traditionally QA is written as a quantum circuit.

As shown in Fig. 1, the general structure of the quantum circuit is based on three quantum operators (superposition, entanglement, and interference) and measurement.

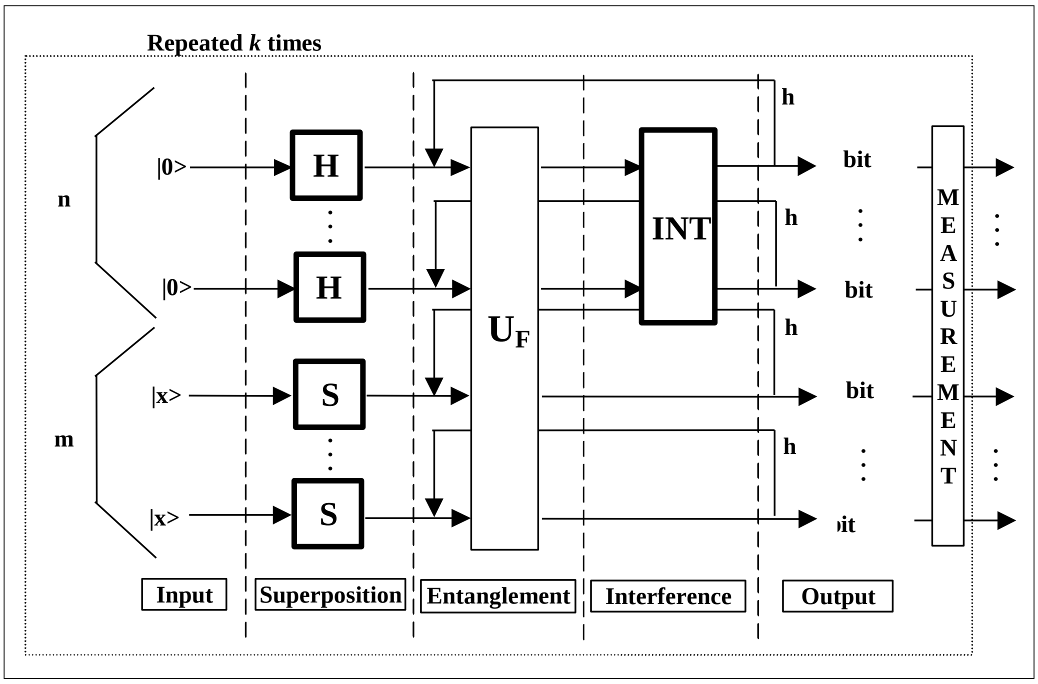


Figure 1. Quantum circuit structure

Input in the quantum circuit acts on an initial canonical basis vector to generate a complex linear combination (called a superposition) of basis vectors as an output. This superposition contains the information to answer the initial problem. After the superposition has been created, measurement takes place in order to extract the answer information. In quantum mechanics, a measurement is a non-deterministic operation that produces as output only one of the basis vectors in the entering superposition.

A general QA, written as a quantum circuit, can be automatically translated into the corresponding programmable quantum gate for efficient classical simulation. This gate is represented as a quantum operator in matrix form such that, when it is applied to the vector input representation of the quantum register state, the result is the vector representation of the desired register output state.

The simulation system of quantum computation is based on QAG's.

The design process of QAG's includes the matrix design form of three quantum operators: superposition *(Sup)*, entanglement (U_F) and interference *(Int)* that are the background of QA structures. In general form, the structure of a QAG can be described as follows (see Chapter 1):

$$QAG = \left[\left(Int \otimes {}^n I\right) \cdot U_F\right]^{h+1} \cdot \left[{}^n H \otimes {}^m S\right], \tag{1}$$

where I is the identity operator; the symbol $\otimes$ denotes the tensor product; S is equal to I or H and dependent on the problem description. One portion of the design process in Eq. (1) is the type-choice of the entanglement problem dependent operator U_F that physically describes the qualitative properties of the function f.

The efficient implementations of a number of operations for quantum computation include controlled phase adjustment of the amplitudes in the superposition, permutation, approximation of transformations and generalizations of the phase adjustments to block matrix transformations. These operations generalize those used as example in quantum search algorithms (QSA's) that can be realized on a classical computer. The application of this approach is applied herein to the efficient simulation on classical computers of the Deutsch QA, the Deutsch–Jozsa QA, the Simon QA, the Shor QA and the Grover QA.

Implementation of a QA is based on a QAG. In the language of classical computing, a quantum computer is programmed by designing a QAG. The prior art reports relatively few such gates because the basic principles underlying the quantum version of programming are in their infancy and algorithms to date have been programmed by ad-hoc techniques.

Fig. 2 is a block diagram showing a gate approach for simulation of a QA using classical computers.

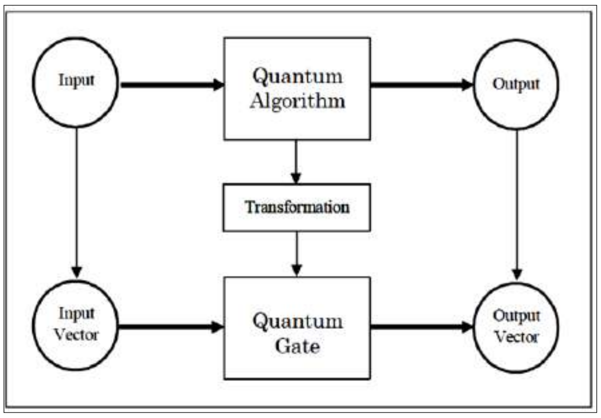


Figure 2. The gate approach for simulation of quantum algorithms using classical computers

In Fig. 2, an input is provided to a QA and the QA produces an output. However, the QA can be transformed to produce a QAG such that an input vector (corresponding to the QA input) is provided to the QAG to produce an output vector (corresponding to the QA output).

Fig. 3 is a block diagram showing the design of the QAG.

In Fig. 3, an input block of the QA is a function f that maps binary strings into binary strings. This function f is represented as a map table block, defined for every string its image. The function is first encoded in corresponding block into a unitary matrix operator U_F depending on the properties of f . In some sense, this operator calculates f when its input and output strings are encoded into canonical basis vectors of a complex Hilbert space.

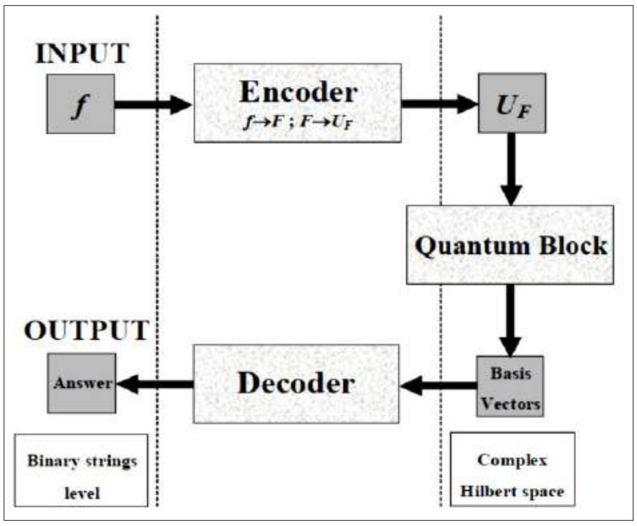


Figure 3. Schematic block diagram of QAG method design

The operator U_F maps the vector code of every string into the vector code of its image by f . The quantum block operates on basis vectors in a complex Hilbert space. The vectors operated on by the quantum block are provided to a decoder, which decodes the vectors to produce an answer. Once generated, the matrix operator U_F is embedded into a quantum gate G.

The quantum gate G is a unitary matrix whose structure depends on the form of matrix U_F and on the problem to be solved. The quantum gate is a unitary operator built from the dot composition of other more specific operators. The specific operators are described as tensor products of smaller matrices.

The quantum circuit is a high-level description of how these smaller matrices are composed using tensor and dot products in order to generate the final quantum gate as shown in Fig. 1.The mathematical background of this approach is based on mappings between the quantum block operations in the complex Hilbert space. The encoder and decoder operate in a map table and interpretation space, and input/output occurs on a binary string level. The Clifford and Pauli groups are the background for universal QAG design for simulation of a QA's on classical computers.

The probability of every basis vector of being the output of measurement depends on its complex coefficient (probability amplitude) in the entering complex linear combination.

Main QAG's and main quantum operators

Three quantum operators, superposition, entanglement, and interference, are the basis for quantum computations of qualitative and quantitative measures in quantum soft computing. As described above, Fig. 3 shows the structure of a QAG based on the three quantum operations of superposition, entanglement, and interference.

Fig. 4 shows methods in QAG design.

The methods as shown in Fig. 4 are based on qualitative measures of QAG design: 1) analysis of QA dynamics and structure gate design; 2) analysis of information flow; and 3) structure simulation of intelligent QA's on classical computers.

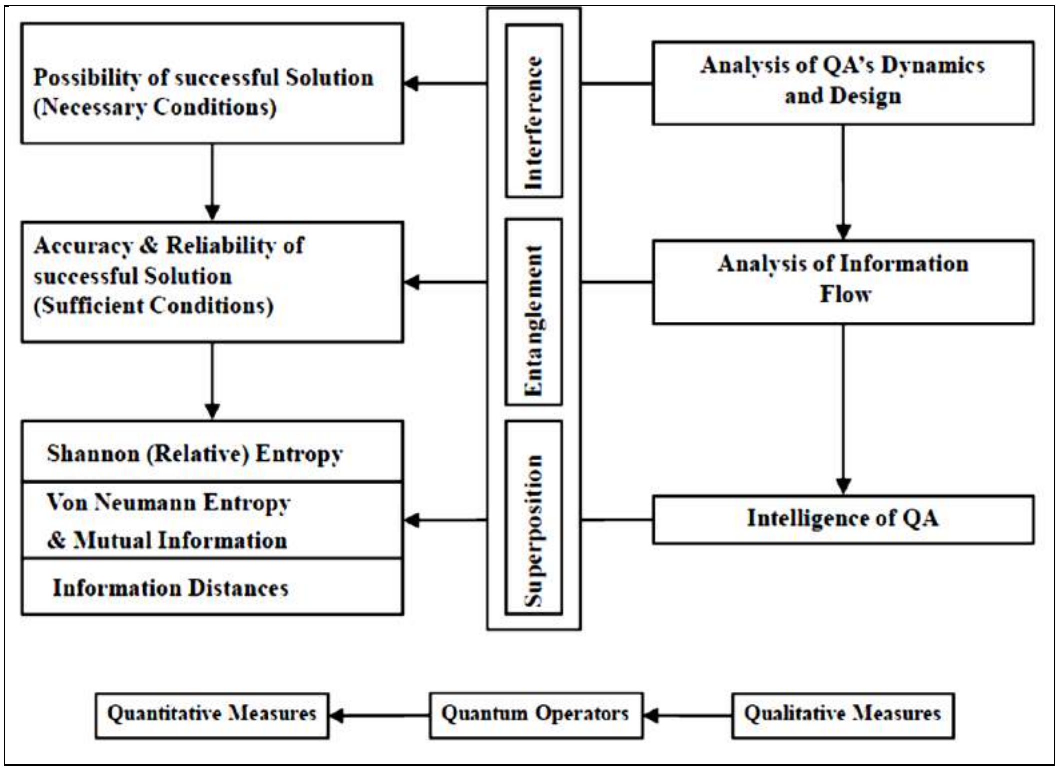

Possibility of successful Solution (Necessary Conditions)
Accuracy & Reliability of successful Solution (Sufficient Conditions)
Shannon (Relative) Entropy
Von Neumann Entropy & Mutual Information
Information Distances
Interference
Entanglement
Superposition
Analysis of QA's Dynamics and Design
Analysis of Information Flow
Intelligence of QA
Quantitative Measures
Quantum Operators
Qualitative Measures

Figure 4. Methods in Quantum Algorithm Gate Design

In this paper analysis of QA dynamics and structure gate design, and structure simulation of intelligent QA's on classical computers are discussed.

As shown in Fig. 4 analysis of QA dynamics provides the background for showing the existence of a solution and that the solution is unique with the desired probability. Analysis of information flow in the QA gates provides the background for showing that the unique solution exists with the desired accuracy and that the reliability of the solution can be achieved with higher probability.

With the method of quantum gate design presented herein, various different structures of QA can be realized, as shown in Table 1 below.

The intelligence of a QA is achieved through the principle of minimum information distance between Shannon and von Neumann entropy and includes the solution of the QA stopping problem.

The output states of a QA as the solution of expected problems are the intelligent states with minimum entropic relations of uncertainty (coherent superposition states). The successful results of QA computing are robust to noise excitations in quantum gates, and intelligent quantum operations are fault-tolerant in quantum soft computing.

Table 1. Quantum gate parameters for QA's structure design

Name	Algorithm	Gate Symbolic Form: $\left[\underbrace{\left(Int \otimes {}^{m}I\right)}_{Interference} \cdot \underbrace{U_F}_{Entanglement}\right]^{h+1} \cdot \left(\underbrace{{}^{n}H \otimes {}^{m}S}_{Superposition}\right)$
Deutsch-Jozsa (D. – J.)	$m=1,\ S=H\ (x=1)$ $Int={}^{n}H$ $k=1\ h=0$	$\left({}^{n}H \otimes I\right)\cdot U_F^{D.-J.}\cdot\left({}^{n+1}H\right)$
Simon (Sim)	$m=n,\ S=I$ $(x=0)\ Int={}^{n}H\ \ k=O(n)$ $h=0$	$\left({}^{n}H \otimes {}^{n}I\right)\cdot U_F^{Sim}\cdot\left({}^{n}H \otimes {}^{n}I\right)$
Shor (Shr)	$m=n,\ \ S=I\ (x=0)$ $Int=QFT_n$ $k=O\left(Poly(n)\right)\ h=0$	$\left(QFT_n \otimes {}^{n}I\right)\cdot U_F^{Shr}\cdot\left({}^{n}H \otimes {}^{n}I\right)$
Grover (Gr)	$m=1,\ \ S=H\ (x=1)$ $Int=D_n$ $k=1,\ \ h=O\left(2^{n/2}\right)$	$\left(D_n \otimes I\right)\cdot U_F^{Gr}\cdot\left({}^{n+1}H\right)$

A quantum computer is difficult to build because of decoherence effects.

Decoherence introduces errors in the superposition.

The decoherence problem is reduced by using tools of quantum soft computing such as a quantum genetic search algorithm (QGSA). Errors produced by decoherence are of three kinds: (i) phase errors; (ii) bit-flip errors; and (iii) both phase and bit-flip errors. These three errors can all be modeled using unitary transformations.

This means that if the QGSA is implemented on a physical quantum-mechanical system, one would gain the advantages of quantum parallelism and reduce the problem of decoherence, because decoherence can be used as a natural generator of mutation and crossover operators.

Design technology of quantum algorithmic gate boxes and simulation system

The problems solved by the QA can be stated as follows:

Input	*A function* $f:\{0,1\}^n \to \{0,1\}^m$
Problem	*Find a certain property of* f

The structure of a quantum operator U_F in QA's as shown in block of Fig. 3 is outlined, with a high level representation, in the scheme diagram Fig. 1. In Fig. 3 the input of the QA is a function *f* that maps from binary strings into binary strings. This function is represented as a map table, defining for every string its image. The function *f* is encoded according to an F -truth table. The function is transformed according to a transform U_F -truth table into a unitary matrix operator U_F depending on *f*'s properties. In some sense, this operator calculates *f* when its input and output strings are encoded into canonical basis vectors of a complex Hilbert space: U_F maps the vector code of every string into the vector code of its image by *f*. A squared matrix U_F on the complex field is unitary if and only if (iff) its inverse matrix coincides with its conjugate transpose: $U_F^{-1} = U_F$. A unitary matrix is always reversible and preserves the norm of vectors.

Fig. 5 shows structure of the quantum block from Fig. 3.

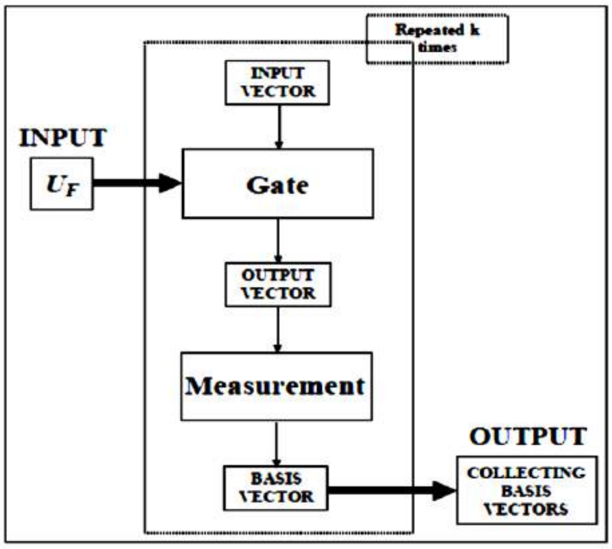


Figure 5. Structure of Quantum Block in Fig. 3

In the structure, the matrix operator U_F has been generated it is embedded into a quantum gate as a QAG, a unitary matrix whose structure depends on the form of matrix U_F and on the problem to be solved. In the QA, the QG acts on an initial canonical basis vector (which can always choose the same vector) in order to generate a complex linear combination (superposition) of basis vectors as output. This superposition contains all the information to answer the initial problem.

After this superposition has been created, in measurement block takes place in order to extract this information. In quantum mechanics, measurement is a non-deterministic operation that produces as output only one of the basis vectors in the entering superposition. The probability of every basis vector of being the output of measurement depends on its complex coefficient (probability amplitude) in the entering complex linear combination.

The segmental action of the QAG and of measurement characterizes the quantum block in Fig. 5. The quantum block is repeated *k* times in order to produce a collection of *k* basis vectors. Since measurement a nondeterministic operation, these basic vectors are not be necessarily identical and each one of them will encode a piece of the information needed to solve the problem. The collection block in Fig. 3.5 of the algorithm outputs the interpretation of the collected basis vectors in order to get the answer for the initial problem with a certain probability.

Encoder

The behavior of the encoder in Fig. 3 is described in the scheme diagram of Fig. 6. Function *f* is encoded into matrix U_F in three steps.

In *step* 1, the map table ($f-$truth table) of function *f*: $\{0,1\}^n \to \{0,1\}^m$ is transformed into the map table ($F-$truth table) of the injective function F:$\{0,1\}^{n+m} \to \{0,1\}^{n+m}$ such that:

$$F(x_0, .., x_{n-1}, y_0, .., y_{m-1}) = (x_0, .., x_{n-1}, f(x_0, .., x_{n-1}) \oplus (y_0, .., y_{m-1})).$$

Remark. The need to deal with an injective function comes from the requirement that U_F is unitary. A unitary operator is reversible, so it cannot map 2 different inputs in the same output. Since U_F will be the matrix representation of F, F is injective. If one directly employed the matrix representation of function f, one could obtain a non-unitary matrix, since f could be non-injective. So, injectivity is fulfilled by increasing the number of bits and considering function F instead of function f. The function f can be calculated from F by putting $(y_0,...,y_{m-1}) = (0,...,0)$ in the input string and reading the last m values of the output string.

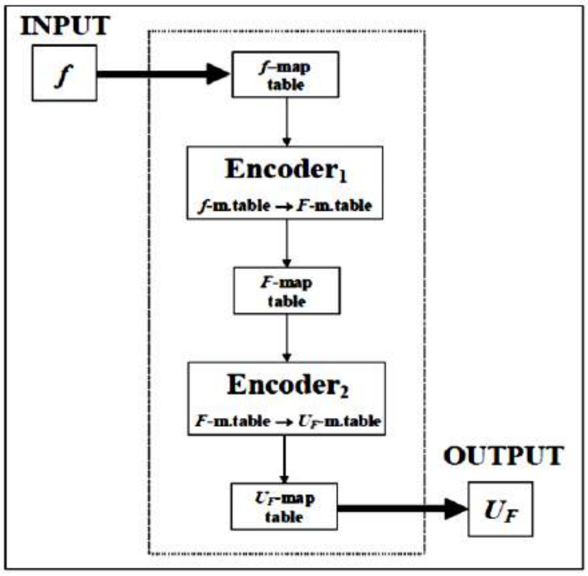


Figure 6. The encoder block scheme diagram

Reversible circuits realize permutation operations. It is possible to realize any Boolean circuit $F:\mathbb{B}^n \to \mathbb{B}^m$ by reversible circuit. For this case, one need not calculate the function $F:\mathbb{B}^n \to \mathbb{B}^m$. One can calculate another function with expanding $F_\oplus:\mathbb{B}^{n+m} \to \mathbb{B}^{n+m}$ that is defined as following relation: $F_\oplus(x,y)=(x, y\oplus F(x))$ where the operation $\oplus$ is defined as addition on module 2.

Then the value of $F(x)$ is defined as $F_\oplus(x,0)=(x,F(x))$. For example, the *XOR* operator between two binary strings p and q of length m is a string s of length m such that the i-th digit of s is calculated as the exclusive *OR* between the i-th digits of p and q:

$$p = (p_0, .., p_{n-1}),\ q = (q_0, .., q_{n-1});\ s = p \oplus q = ((p_0+q_0) \bmod 2, .., (p_{n-1}+q_{n-1}) \bmod 2)).$$

In *step* 2, the function from F map table is transformed into U_F map table, according to the following constraint:

$$\forall s\in\{0,1\}^{n+m} : U_F[\tau(s)] = \tau[F(s)] \qquad (2)$$

The code map τ: $\{0,1\}^{n+m} \to \mathrm{C}^{2^{n+m}}$ ($\mathrm{C}^{2^{n+m}}$ is the target Complex Hilbert Space) is such that:

$$\tau(0)=\begin{pmatrix}1\\0\end{pmatrix}=|0\rangle, \qquad \tau(1)=\begin{pmatrix}0\\1\end{pmatrix}=|1\rangle$$
$$\tau(x_0,\dots,x_{n+m-1})=\tau(x_0)\otimes\dots\otimes\tau(x_{n+m-1})=|x_0\dots x_{n+m-1}\rangle .$$

Code τ maps bit values into complex vectors of dimension 2 belonging to the canonical basis of C^2. Besides, using tensor product, τ maps the general state of a binary string of dimension n into a vector of dimension 2^n, reducing this state to the joint state of the n bits composing the register. Every bit state is transformed into the corresponding 2-dimesional basis vector and then the string state is mapped into the corresponding 2^n-dimesional basis vector by composing all bit-vectors through tensor product. In this sense tensor product is the vector counterpart of state conjunction.

The tensor product between two vectors of dimensions *h* and *k* is a tensor product of dimension *h*·*k*, such that:

$$\begin{pmatrix} x_1 \\ \dots \\ x_h \end{pmatrix} \otimes \begin{pmatrix} y_1 \\ \dots \\ y_k \end{pmatrix} = \begin{pmatrix} x_1\, y_1 \\ \dots \\ x_1\, y_k \\ \dots \\ x_h\, y_1 \\ \dots \\ x_h\, y_k \end{pmatrix}.$$

If a component of a complex vector is interpreted as the probability amplitude of a system of being in a given state (indexed by the component number), the tensor product between two vectors describes the joint probability amplitude of two systems of being in a joint state.

For example:

$$(0,0) \xrightarrow{\tau} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle, \quad (0,1) \xrightarrow{\tau} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle,$$

$$(1,0) \xrightarrow{\tau} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |10\rangle, \quad (1,1) \xrightarrow{\tau} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |11\rangle.$$

Basis vectors are denoted using the *ket* notation $|i\rangle$. This notation is taken from Dirac description of quantum mechanics.

In *step* 3, the U_F map table is transformed into U_F using the following transformation rule:

$$[U_F]_{ij} = 1 \Leftrightarrow U_F\,|j\rangle = |i\rangle\,.$$

This rule can be understood by considering vectors $|i\rangle$ and $|j\rangle$ as column vectors. These vectors belong to the canonical basis, where U_F defines a permutation map of the identity matrix rows. In general, row $|j\rangle$ is mapped into row $|i\rangle$.

This rule will be illustrated in detail below, in the example based on Deutsch's algorithm.

Quantum block

The heart of the quantum block is the quantum gate, which depends on the properties of matrix U_F. The quantum block uses the QAG, which depends on the properties of matrix U_F. The structure of a quantum operator U_F in QA's as shown in Fig. 3 is outlined, with a high level representation, in the scheme diagram of Fig. 5.

The scheme in Fig. 5 gives a more detailed description of the quantum block. The matrix operator U_F of Fig. 6 is the output of the encoder block represented in Fig. 3.

Here, it becomes the input for the quantum block. This matrix operator is embedded into a more complex gate: the gate G (QAG). Unitary matrix G is applied k times to an initial canonical basis vector $|i\rangle$ of dimension 2^{n+m}. Each time, the resulting complex superposition $\mathrm{G}|0\ldots01\ldots1\rangle$ of basis vectors is measured in measurement block, producing one basis vector $|x_i\rangle$ as result. The measured basis vectors $\{x_1,\ldots,x_k\}$ are collected together in block of basis vectors.

This collection is the output of the quantum block. The "intelligence" of the QA's is in the ability to build a QAG that is able to extract the information necessary to find the required property of f and to store it into the output vector collection.

In order to represent QAGs it is useful to employ some diagrams called quantum circuits, as shown in Fig. 1. Each rectangle is associated with a matrix $2^n\times 2^n$, where n is the number of lines entering and leaving the rectangle. For example, the rectangle marked U_F is associated with the matrix U_F.

Using a high-level description of the gate and, using transformation rules shown in Fig. 7, it is possible to compile the corresponding gate-matrix.

These rules are listed in Fig. 7 as following: (a) *Rule* 1 – Tensor Product Transformation; (b) *Rule* 2 – Dot Product Transformation; (c) *Rule* 3 – Identity Transformation; (d) *Rule* 4 – Propagation Rule; (e) *Rule* 5 – Iteration Rule; and (f) *Rule* 6 – Input/Output Tensor Rule.

It will be clearer how to use these rules when we afford the first examples of quantum algorithm.

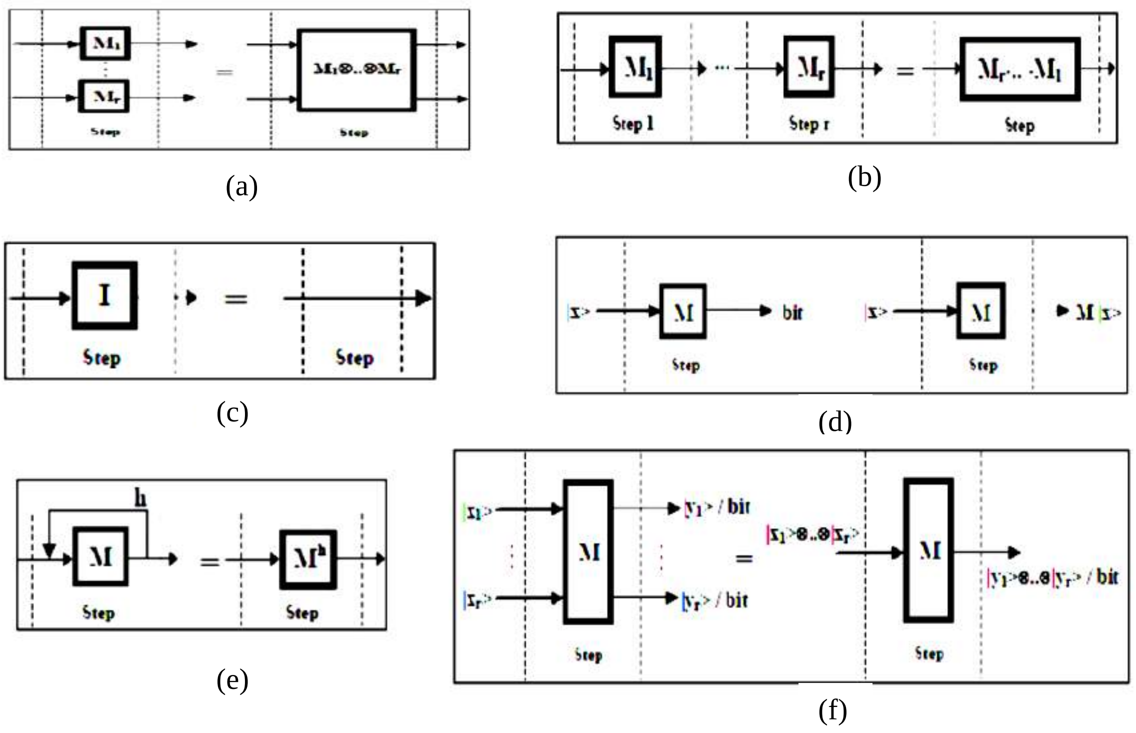


Figure 7. Transformation rules

The tensor product between two matrices $X_{n\times m}$ and $Y_{h\times k}$ is a (block) matrix $(n\cdot h)\times(m\cdot k)$ such that:

$$X\otimes Y=\begin{bmatrix} x_{11}Y & .. & x_{1m}Y \\ .. & .. & .. \\ x_{n1}Y & .. & x_{nm}Y \end{bmatrix} \quad with \quad X=\begin{bmatrix} x_{11} & .. & x_{1m} \\ .. & .. & .. \\ x_{n1} & .. & x_{nm} \end{bmatrix}.$$

An example of a matrix tensor product is as follows:

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\otimes\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}=\begin{bmatrix}1\cdot\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix} & 2\cdot\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}\\3\cdot\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix} & 4\cdot\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}\end{bmatrix}=\begin{bmatrix}5 & 6 & 10 & 12\\7 & 8 & 14 & 16\\15 & 18 & 20 & 24\\21 & 24 & 28 & 32\end{bmatrix}.$$

Decoder

The decoder block of Fig. 3 interprets the basis vectors (collected in block basis vectors) of after the iterated execution in the quantum block. Decoding these vectors involves retranslating them into binary strings and interpreting them directly in decoder block if they already contain the answer or use them, for instance as coefficients vectors for some equation system, in order to get the searched solution.

Examples of design method application: QA's Benchmark's gate design and simulation of decision making QA

Let us consider Benchmarks of QAG design for typical QA.

Deutsch's algorithm

In order to illustrate the general method to synthesize a QA and the QG implementing it, a simple pedagogical example, Deutsch's algorithm, is used. The roles of superposition, entanglement and parallel quantum massive calculation are illustrated by this example.

Deutsch's problem: A function f:{0,1}→{0,1} is said constant iff $\exists y\in\{0,1\}:\forall x\in\{0,1\}: f(x)=y$. It is said to be balanced iff $|\{x\in\{0,1\}: f(x)=0\}| = |\{x\in\{0,1\}: f(x)=1\}|$.

Thus, Deutsch's problem can be stated as follows:

Input	*A balanced or constant function f*
Problem	*Decide if f is constant or balanced*

Figure 8 shows the structure of Deutsch's problem.

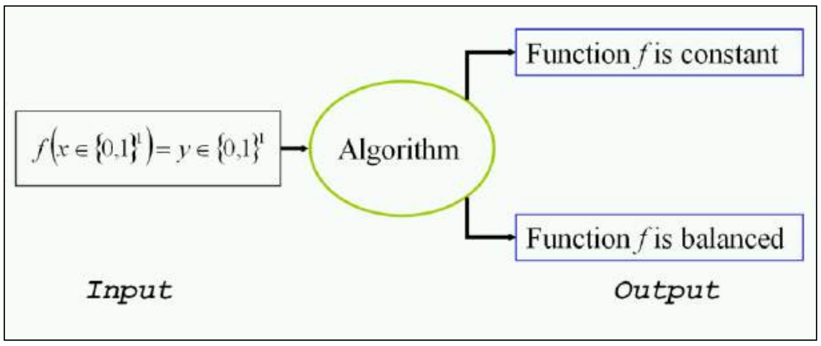


Figure 8. Problem definition of Deutsch's QA

There are four possible functions f_i: {0,1}→{0,1}.

They are defined by the following map tables:

Constant Functions	**Balanced Functions**

	x	$f_1(x)$
(1)	0	0
	1	0

	x	$f_2(x)$
(2)	0	1
	1	1

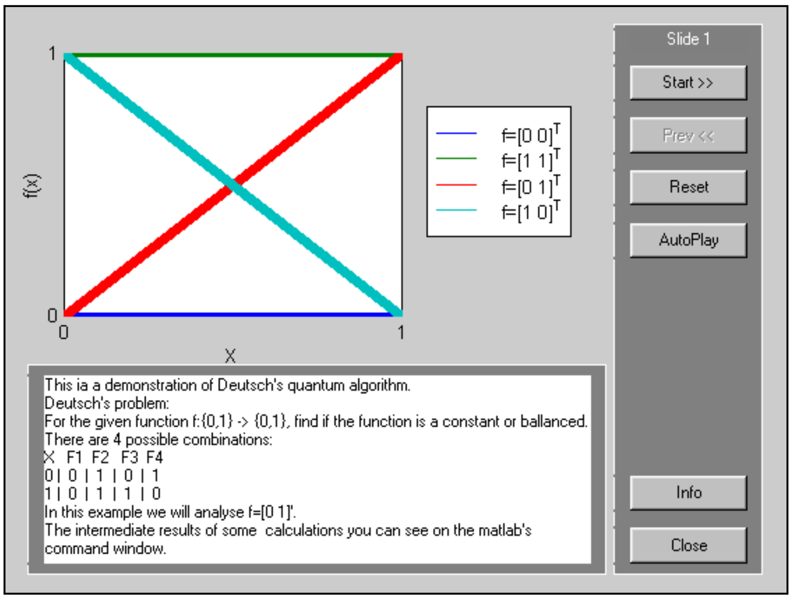
	x	$f_3(x)$
(3)	0	0
	1	1

	x	$f_4(x)$
(4)	0	1
	1	0

The set $\{f_i\}_{i\in\{1,2,3,4\}}$ is the input set for our algorithm.

Every function f_i is represented by its map table.

Fig. 9 shows definitions of constant and balanced functions.



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This ia a demonstration of Deutsch's quantum algorithm.
Deutsch's problem:
For the given function f:{0,1} -> {0,1}, find if the function is a constant or ballanced.
There are 4 possible combinations:
X  F1  F2  F3  F4
0|  0 | 1 | 0 | 1
1|  0 | 1 | 1 | 0
In this example we will analyse f=[0 1]'.
The intermediate results of some  calculations you can see on the matlab's
command window.
```

Figure 9. Deutsch's quantum algorithm simulation: Problem definition visualization

Encoder. The encoder block encodes input function f into matrix U_F. If, for example, the function to be investigated is $f = f_3$, then the map table is the following:

x	$f_3(x)$
0	0
1	1

Step 1

Function f is first transformed into function F:$\{0,1\}^2 \to \{0,1\}^2$ such that

$$F(x_0, y_0) = (x_0, f(x_0)\oplus y_0).$$

In logic representation this means:

y_0	$F(x_0, y_0)$
0	$(x_0, f(x_0))$
1	$(x_0, \neg f(x_0))$

As usual, the NOT operator acting on a binary string flips the value of every digit in the string

$$p = (p_0, \ldots, p_{n-1}),\ \neg p = ((p_0+1)\mathrm{mod}2, \ldots, (p_{n-1}+1)\mathrm{mod}2).$$

Therefore, if $f = f_3$, F- map table is the following:

(x_0, y_0)	**$F(x_0, y_0)$**
(0,0)	(0,0)
(0,1)	(0,1)
(1,0)	(1,1)
(1,1)	(1,0)

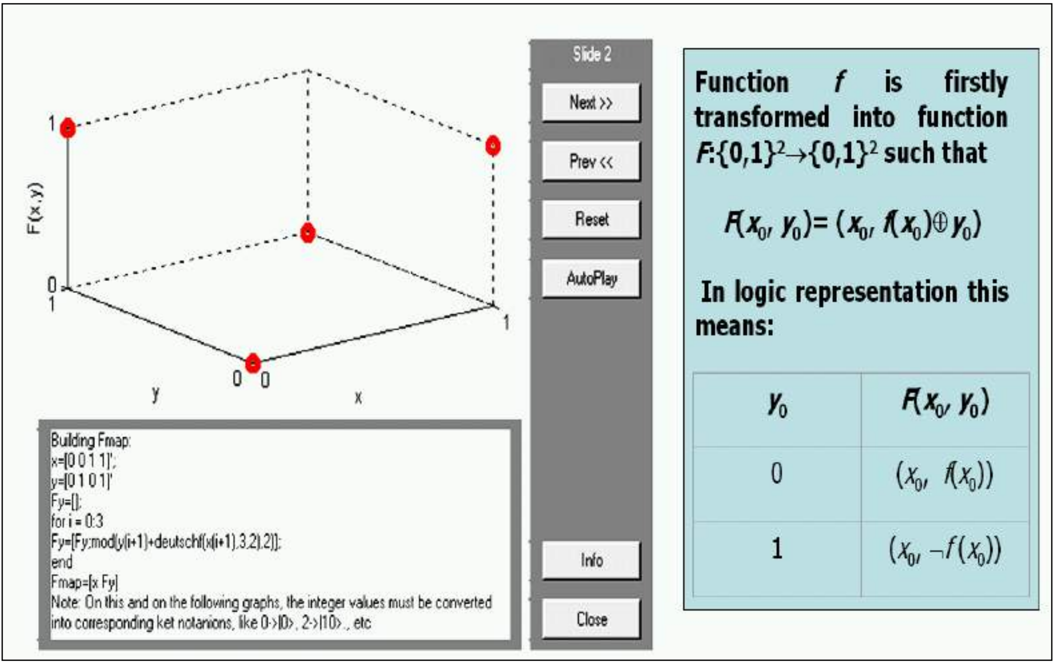
Fig. 10 shows the result of F-map table building.

Step 2

In this step, the map table of F is transformed into the map table of U_F.

The transformation rule is the following:

$$\forall s \in \{0,1\}^2 \colon U_F[\tau(s)] = \tau[F(s)].$$



```
Building Fmap:
x=[0 0 1 1]';
y=[0 1 0 1]'
Fy=[];
for i = 0:3
Fy=[Fy;mod(y(i+1)+deutschf(x(i+1),3,2),2)];
end
Fmap=[x Fy]
Note: On this and on the following graphs, the integer values must be converted
into corresponding ket notanions, like 0->|0> , 2->|10> ., etc
```

Function f is firstly transformed into function F:$\{0,1\}^2 \to \{0,1\}^2$ such that

$$F(x_0, y_0) = (x_0, f(x_0) \oplus y_0)$$

In logic representation this means:

y_0	$F(x_0, y_0)$
0	$(x_0, f(x_0))$
1	$(x_0, \neg f(x_0))$

Figure 10. Deutsch's quantum algorithm simulation, Step 1: F-map table building

So, U_F map table is:

$\lvert x_0 y_0 \rangle$	**U_F** $\lvert x_0 y_0 \rangle$
$\lvert 00 \rangle$	$\lvert 00 \rangle$
$\lvert 01 \rangle$	$\lvert 01 \rangle$
$\lvert 10 \rangle$	$\lvert 11 \rangle$
$\lvert 11 \rangle$	$\lvert 10 \rangle$

or, writing basis vectors as column vectors:

v	***U_F v***
$(1,0,0,0)^T$	$(1,0,0,0)^T$
$(0,1,0,0)^T$	$(0,1,0,0)^T$
$(0,0,1,0)^T$	$(0,0,0,1)^T$
$(0,0,0,1)^T$	$(0,0,1,0)^T$

The *TRANSPOSE* (*T*) operator acting on a row or column vector transforms the vector into its corresponding column or, row vector (respectively):

$$\begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}^T = \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}; \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}^T = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}.$$

Step 3

The matrix associated with such a map table is obtained from the identity matrix 4×4 by a permutation of its rows: the first and the second rows are mapped into themselves, whereas the third row is mapped into the fourth one and the fourth row into the third one:

$$U_F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

A general way to build U_F is to express every vector U_F ($|s\rangle$) as a linear combination of the basis vectors. The coordinates of this combination are all 0, unless for one basis vector corresponding to the image of $|s\rangle$ by U_F:

$$\begin{aligned} U_F|00\rangle &= 1|00\rangle + 0|01\rangle + 0|10\rangle + 0|11\rangle \\ U_F|01\rangle &= 0|00\rangle + 1|01\rangle + 0|10\rangle + 0|11\rangle \\ U_F|10\rangle &= 0|00\rangle + 0|01\rangle + 0|10\rangle + 1|11\rangle \\ U_F|11\rangle &= 0|00\rangle + 0|01\rangle + 1|10\rangle + 0|11\rangle \end{aligned}.$$

Calculate $[U_F]_{ij}$ as the coordinate of vector U_F ($|j\rangle$) with respect to vector $|i\rangle$, where *i* and *j* are binary sequences. This means:

$$[U_F]_{ij} = 1 \Leftrightarrow U_F|j\rangle = |i\rangle.$$

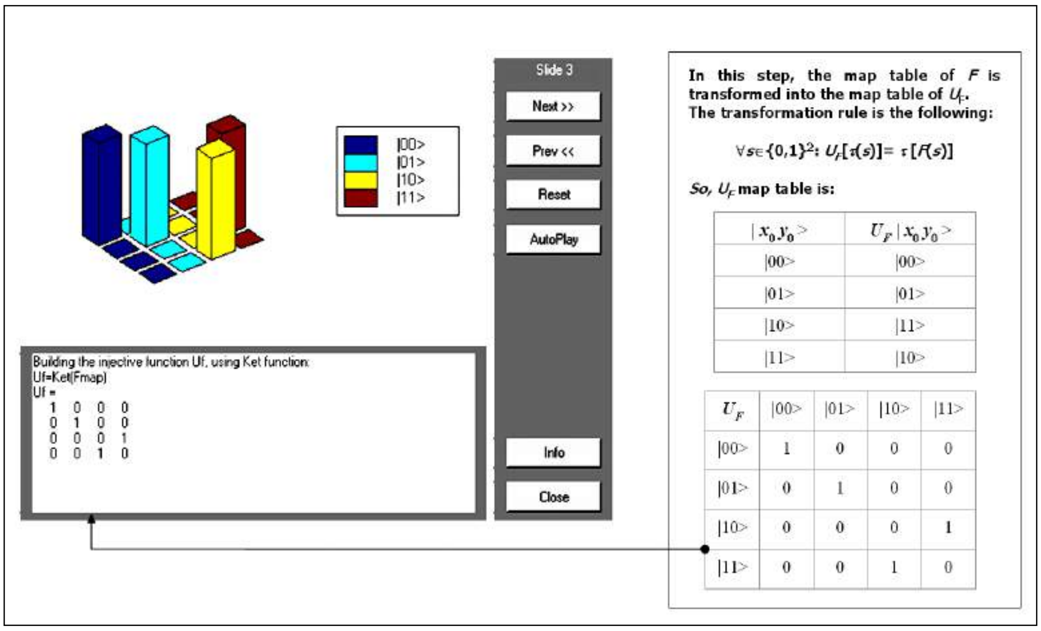
Value $[U_F]_{\mathrm{ij}}$ is called the probability amplitude of $|j\rangle$ being mapped $|i\rangle$ into by U_F.

The probability amplitude of $|00\rangle$ of being mapped into $|00\rangle$ is, for instance, 1, since $U_F|00\rangle=1|00\rangle$, whereas its probability amplitude of being mapped into $|01\rangle$ is 0, since $U_F|00\rangle=0|01\rangle$. Using this technique, the following unitary matrix is built:

U_F	$\|00\rangle$	$\|01\rangle$	$\|\|10\rangle$	$\|11\rangle$
$\|00\rangle$	1	0	0	0
$\|01\rangle$	0	1	0	0

$\|10\rangle$	0	0	0	1
$\|11\rangle$	0	0	1	0

Fig. 11 shows the design process of unitary matrix U_F.



```
Building the injective function Uf, using Ket function:
Uf=Ket(Fmap)
Uf =
    1   0   0   0
    0   1   0   0
    0   0   0   1
    0   0   1   0
```

In this step, the map table of F is transformed into the map table of U_F. The transformation rule is the following:

$\forall s \in \{0,1\}^2$: $U_F[\tau(s)] = \tau[F(s)]$

So, U_F map table is:

$\|x_0 y_0>$	$U_F\|x_0 y_0>$
|00>	|00>
|01>	|01>
|10>	|11>
|11>	|10>

U_F	|00>	|01>	|10>	|11>
|00>	1	0	0	0
|01>	0	1	0	0
|10>	0	0	0	1
|11>	0	0	1	0

Figure 11. Deutsch's quantum algorithm simulation, Step 2: Entanglement operator

Quantum block. The encoder block has generated matrix U_F. This matrix is now embedded into the QG that will act on the input vector $|00\rangle$.

Fig. 12 (a) shows this gate using a quantum circuit.

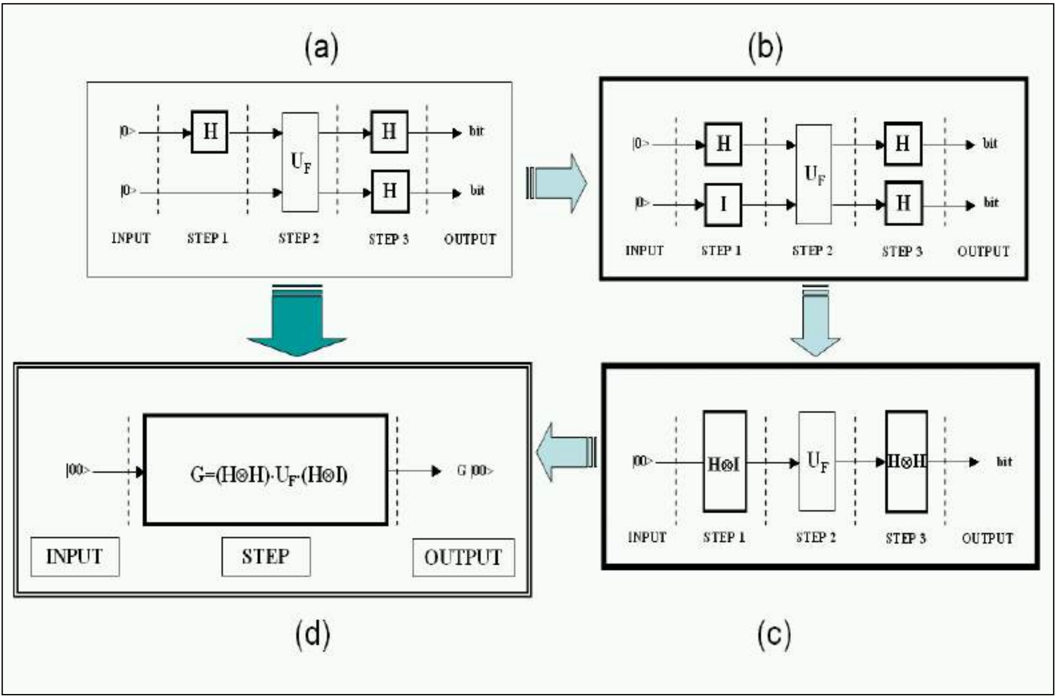


Figure 12. Deutsch's quantum algorithm simulation:

Circuit representation and corresponding gate design

Each rectangle in Fig. 12 (a) represents a classical matrix operator *n*×*n*, where *n* is the number of lines entering and leaving the rectangle.

A matrix operator is said classical, when it maps every basis vector into another basis vector. For example, operator U_F is classical. A thick rectangle stands for a non-classical matrix operator. A non-classical matrix operator maps at least one basis vector into a superposition of basis vectors.

Example: Classical and Non-Classical Matrix operators.

Classical Matrix Operator U_F

U_F	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|11\rangle$
$\|00\rangle$	1	0	0	0
$\|01\rangle$	0	1	0	0
$\|10\rangle$	0	0	0	1
$\|11\rangle$	0	0	1	0

Non-Classical Matrix Operator H

H	$\|0\rangle$	$\|1\rangle$
$\|0\rangle$	$1/2^{1/2}$	$1/2^{1/2}$
$\|1\rangle$	$1/2^{1/2}$	$-1/2^{1/2}$

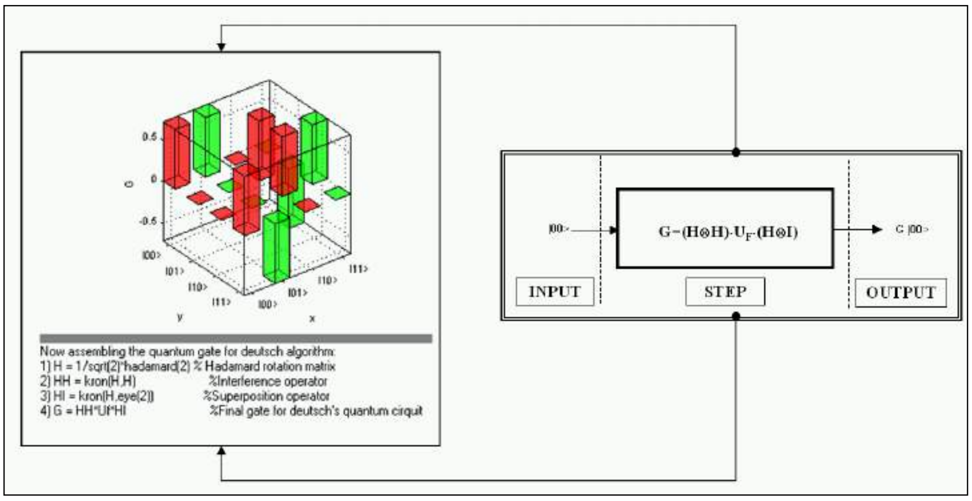
The above circuit is compiled into the corresponding computable gate. The first passage involves completing the circuit making some operators explicit. Consider, for instance, Step 1 in Fig. 12 (a). The second line is empty in this step. This means that the second entering basis vector is left unchanged. This vector acts in the identity matrix operator and completes the circuit. This is rule 3 described in Fig. 7. The result of the compilation is presented on the Fig. 12 (b). The identity matrix operator is classical and it is so defined as:

I	$\|0\rangle$	$\|1\rangle$
$\|0\rangle$	1	0
$\|1\rangle$	0	1

At this point a matrix operator is built corresponding to every step whose action corresponds to the concurrent action of the matrix operators acting on parallel lines. Rules 1 and 6 from Fig. 7 are used to obtain as the quantum circuit of Fig. 12 (c).

Finally, unique matrix operator is built that is equivalent to the sequential application of the operators in step 1, step 2 and step 3. This is operator composition and it is obtained with the dot product among matrices in the reverse order of application, as rule 2 states. Applying rule 2 from the Fig. 7 to the circuit yields as the quantum circuit of Fig. 12 (d), namely the programmable gate implementing Deutsch's algorithm.

Figure 13 shows the result of computer design of QG of Deutsch's QA.



```
Now assembling the quantum gate for deutsch algorithm:
1) H = 1/sqrt(2)*hadamard(2) % Hadamard rotation matrix
2) HH = kron(H,H)             %Interference operator
3) HI = kron(H,eye(2))       %Superposition operator
4) G = HH*Uf*HI               %Final gate for deutsch's quantum cirquit
```

Figure 13. Deutsch's quantum algorithm simulation, Step 4: Quantum gate assembling

Computational steps of design process. To compute and design the gate, first calculate ($H \otimes I$). The output matrix is 4×4. Label each column and row with the corresponding basis vector. Calculate the amplitude probability for each basis vector of being mapped into another basis vector using H and I. Take vector |00> for instance: its probability amplitude of being transformed into |01> is the product between the probability amplitude of |0> of being mapped into |0> by H and the probability amplitude of |0> of being transformed into |1> by I. This is the tensor product.

Therefore:

H	|0>	|1>
|0>	$1/2^{1/2}$	$1/2^{1/2}$
|1>	$1/2^{1/2}$	$-1/2^{1/2}$

and

I	|0>	|1>
|0>	1	0
|1>	0	1

The values $H \otimes I$ and $H \otimes H$ are calculated as follows:

$H \otimes I$	|00>	|01>	|10>	|11>
|00>	$1/2^{1/2}$	0	$1/2^{1/2}$	0
|01>	0	$1/2^{1/2}$	0	$1/2^{1/2}$
|10>	$1/2^{1/2}$	0	$-1/2^{1/2}$	0
|11>	0	$1/2^{1/2}$	0	$-1/2^{1/2}$

$H \otimes H$	|00>	|01>	|10>	|11>
|00>	1/2	1/2	1/2	1/2
|01>	1/2	-1/2	1/2	-1/2
|10>	1/2	1/2	-1/2	-1/2
|11>	1/2	-1/2	-1/2	1/2

One can rewrite U_F when $f = f_3$:

U_{F_3}	|00>	|01>	|10>	|11>
|00>	1	0	0	0
|01>	0	1	0	0
|10>	0	0	0	1
|11>	0	0	1	0

The final programmable gate $G_3 = (H \otimes H)\cdot(U_{F_3}\cdot(H \otimes I))$ is obtained as:

$U_{F_3}\cdot(H\otimes I)$	|00>	|01>	|10>	|11>
|00>	$1/2^{1/2}$	0	$1/2^{1/2}$	0
|01>	0	$1/2^{1/2}$	0	$1/2^{1/2}$
|10>	0	$1/2^{1/2}$	0	$-1/2^{1/2}$
|11>	$1/2^{1/2}$	0	$-1/2^{1/2}$	0

G_3	|00>	|01>	|10>	|11>
|00>	$1/2^{1/2}$	$1/2^{1/2}$	0	0
|01>	0	0	$1/2^{1/2}$	$-1/2^{1/2}$
|10>	0	0	$1/2^{1/2}$	$1/2^{1/2}$
|11>	$1/2^{1/2}$	$-1/2^{1/2}$	0	0

To calculate the programmable gates for the other possible input functions, the map tables are as follows:

x	***f*₁(*x*)**
0	0
1	0

(*x*₀, *y*₀)	***F*₁(*x*₀, *y*₀)**
(0,0)	(0,0)
(0,1)	(0,1)
(1,0)	(1,0)
(1,1)	(1,1)

x	***f*₂(*x*)**
0	1
1	1

(*x*₀, *y*₀)	***F*₂(*x*₀, *y*₀)**
(0,0)	(0,1)
(0,1)	(0,0)
(1,0)	(1,1)
(1,1)	(1,0)

From every table, it is easy to calculate the matrix operator:

|*x*₀ *y*₀>	**U_{F_1} |*x*₀ *y*₀>**
|00>	|00>
|01>	|01>
|10>	|10>
|11>	|11>

U_{F_1}	|00>	|01>	|10>	|11>
|00>	1	0	0	0
|01>	0	1	0	0
|10>	0	0	1	0
|11>	0	0	0	1

$\|x_0\, y_0>$	$U_{F_2}\, \|x_0\, y_0>$
|00>	|01>
|01>	|00>
|10>	|11>
|11>	|10>

U_{F_2}	|00>	|01>	|10>	|11>
|00>	0	1	0	0
|01>	1	0	0	0
|10>	0	0	0	1
|11>	0	0	1	0

(x_0, y_0)	$U_{F_4}\, \|x_0\, y_0>$
|00>	|01>
|01>	|00>
|10>	|10>
|11>	|11>

U_{F_4}	|00>	|01>	|10>	|11>
|00>	0	1	0	0
|01>	1	0	0	0
|10>	0	0	1	0
|11>	0	0	0	1

Different U_{F_i} (i=1,2,4) generate different programmable gates $G_i=(H \otimes H)\cdot U_{F_i}\cdot(H \otimes I)$:

G_1	|00>	|01>	|10>	|11>
|00>	$1/2^{1/2}$	$1/2^{1/2}$	0	0
|01>	$1/2^{1/2}$	$-1/2^{1/2}$	0	0
|10>	0	0	$1/2^{1/2}$	$1/2^{1/2}$
|11>	0	0	$1/2^{1/2}$	$-1/2^{1/2}$

G_2	|00>	|01>	|10>	|11>
|00>	$1/2^{1/2}$	$1/2^{1/2}$	0	0
|01>	$-1/2^{1/2}$	$1/2^{1/2}$	0	0
|10>	0	0	$1/2^{1/2}$	$1/2^{1/2}$
|11>	0	0	$-1/2^{1/2}$	$1/2^{1/2}$

G_4	|00>	|01>	|10>	|11>
|00>	$1/2^{1/2}$	$1/2^{1/2}$	0	0
|01>	0	0	$-1/2^{1/2}$	$1/2^{1/2}$
|10>	0	0	$1/2^{1/2}$	$1/2^{1/2}$
|11>	$-1/2^{1/2}$	$1/2^{1/2}$	0	0

Finally, different programmable gates generate different superposition states:

G_1|00>	=	$1/2^{1/2}$	|00>	+	$1/2^{1/2}$	|01>
G_2|00>	=	$1/2^{1/2}$	|00>	–	$1/2^{1/2}$	|01>
G_3|00>	=	$1/2^{1/2}$	|00>	+	$1/2^{1/2}$	|11>
G_4|00>	=	$1/2^{1/2}$	|00>	–	$1/2^{1/2}$	|11>

Observe that $G_1|00>$ and $G_2|00>$ can be written as the tensor products of two simpler vectors:

$G_1|00>$	=	$1/2^{1/2}$	$|0>$	$\otimes$	(	$|0>$	+	$|1>$	)
$G_2|00>$	=	$1/2^{1/2}$	$|0>$	$\otimes$	(	$|0>$	–	$|1>$	)

This is not possible for $G_3|00>$ and $G_4|00>$. These two vectors make two entangled states.

This means that Deutsch's QA *needs entanglement* for speed-up of quantum parallel massive calculations.

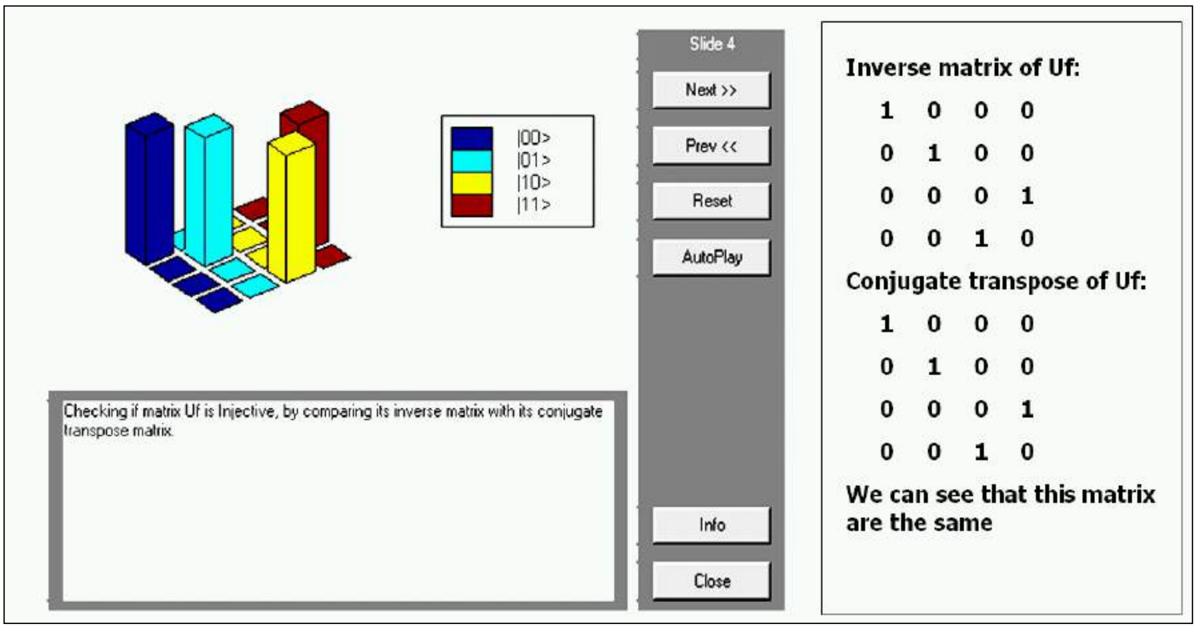
A vector $\underline{v}$ of dimension 2^n is said to represent an entangled state if and only if it cannot be written as the tensor product of *n* vectors of dimension 2. Mathematically, the entanglement condition is written as:

$$\neg\exists \underline{v}_1,\ldots,\underline{v}_n : \underline{v} = \underline{v}_1 \otimes \ldots \otimes \underline{v}_n$$

Figs 14 and 11 show the result of computer check of entanglement property.

When the QAG has generated the output vector, which is a linear complex superposition of basis vectors measurement takes place. It is assumed that measurement is a non-deterministic operation whose input is the linear superposition of basis vectors and whose output is only one of these basis vectors. The probability of a basis vector being the result of measurement is given by the squared modulus of its complex coordinate in the starting superposition.

This description of measurement is taken from quantum mechanics and it is the main constraint on the access one has to the results of the QAG. The non-deterministic evolution of a quantum system by measurement is the true qualitative difference between a quantum computation and a simple parallel computation.



Checking if matrix Uf is Injective, by comparing its inverse matrix with its conjugate transpose matrix.

Inverse matrix of Uf:

```
1  0  0  0
0  1  0  0
0  0  0  1
0  0  1  0
```

Conjugate transpose of Uf:

```
1  0  0  0
0  1  0  0
0  0  0  1
0  0  1  0
```

We can see that this matrix are the same

Figure 14. Deutsch's quantum algorithm simulation, Step 3: checking if entanglement operator is injective or not

In quantum mechanics measurement is a non-deterministic operator. Writing a vector *v* as the complex linear combination of *n* basis vector $v_i, i = 1,\ldots,n$ the probability to observe v_i when *v* is measured is given by the squared modulus of the complex co-ordinate of v_i in *v*.

$$\underline{v} = \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \ldots + \alpha_n \underline{v}_n \xrightarrow{Measurement}$$

Vector	Probability
$\underline{v}_1$	$\lVert\alpha_1\rVert^2$
$\underline{v}_2$	$\lVert\alpha_2\rVert^2$
$\vdots$	$\vdots$
$\underline{v}_n$	$\lVert\alpha_n\rVert^2$

When applying measurement to the superposition of basis vectors resulting from one of our 4 gates, the following is obtained:

Superposition of basis vectors (*before a measurement*)	Result of measurement	
	Vector	*Probability*
G_1|00>=1/√2 |00> + 1/√2 |01>	|00>	$\|\|1/\sqrt{2}\|\|^2=0.5$
	|01>	$\|\|1/\sqrt{2}\|\|^2=0.5$
G_2|00>=1/√2 |00> – 1/√2 |01>	|00>	$\|\|1/\sqrt{2}\|\|^2=0.5$
	|01>	$\|\|1/\sqrt{2}\|\|^2=0.5$
G_3|00>=1/√2 |00> + 1/√2 |11>	|00>	$\|\|1/\sqrt{2}\|\|^2=0.5$
	|11>	$\|\|1/\sqrt{2}\|\|^2=0.5$
G_4|00>=1/√2 |00> – 1/√2 |11>	|00>	$\|\|1/\sqrt{2}\|\|^2=0.5$
	|11>	$\|\|1/\sqrt{2}\|\|^2=0.5$

With measurement, the quantum block ends. In Deutsch's algorithm the quantum block is repeated only one time, so only one resulting basis vector is collected. Thus for success result of decision making is enough 50% of probability.

Decoder. When the final basis vector has been produced, it is interpreted to find the information it carries in order to establish if f is constant or balanced. If the resulting vector is |00> nothing can be said about which function was encoded in U_F. But if the result is |01> or |11>, the function was f_1 or f_2 in the first case, f_3 or f_4 in the second. In fact only gates G_1 and G_2 may produce a vector such that, when it is measured, basis vector |01> has a non-null probability of being observed. Similarly, only gates G_3 and G_4 may produce a superposition of basis vectors where vector |11> has non-null probability amplitude. Since f_1 and f_2 are constant, whereas f_3 and f_4 are balanced, the resulting vector is easily decoded in order to answer Deutsch's problem:

Resulting vector (*after measurement*)	Answer
|00>	Nothing can be said
|01>	*f* is constant
|11>	*f* is balanced

The above described design and calculation processes of QAG for Deutsch's QA can be efficiently simulated on computers with Von Neumann architecture.

Computer design process of Deutsch's QAG and simulation results. Fig. 15 shows the result of computer design of superposition and interference quantum operators.

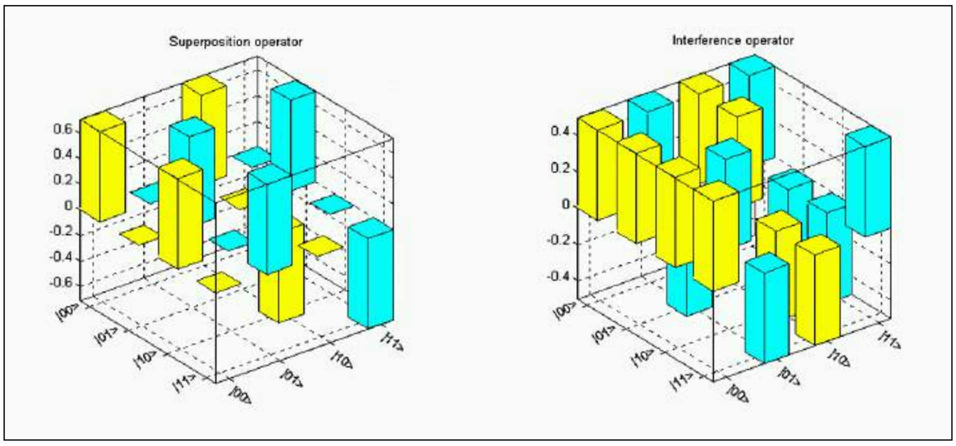
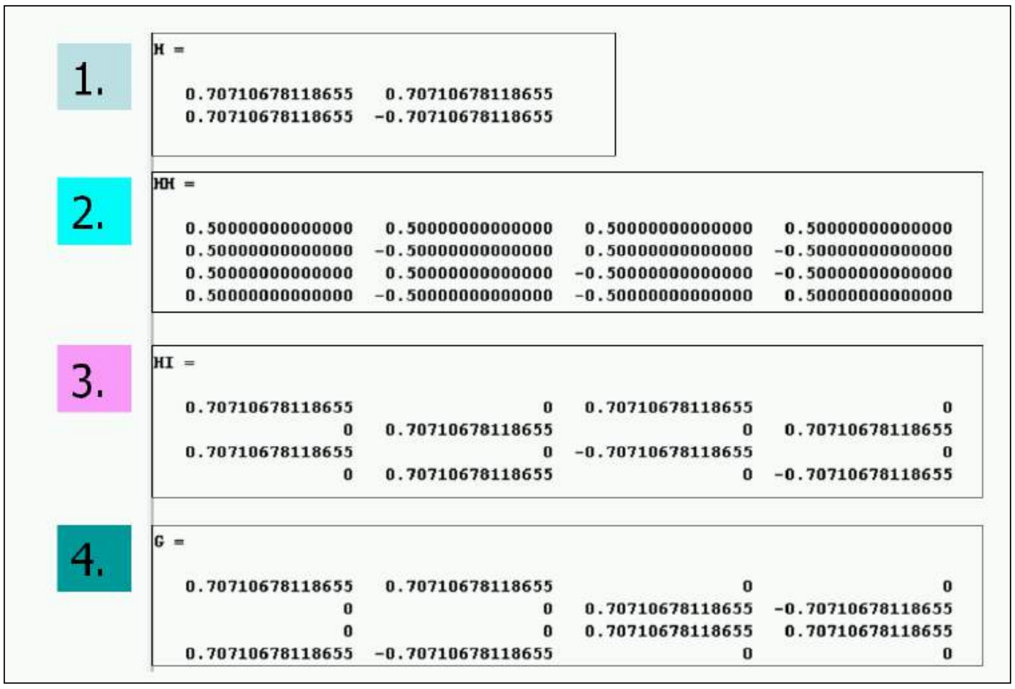


Figure 15. Deutsch’s quantum algorithm: Superposition and Interference operators

Fig. 16 shows the result of QG assembly and results of numerical data simulation using this QAG.



```
1.  H =

      0.70710678118655   0.70710678118655
      0.70710678118655  -0.70710678118655

2.  HH =

      0.50000000000000   0.50000000000000   0.50000000000000   0.50000000000000
      0.50000000000000  -0.50000000000000   0.50000000000000  -0.50000000000000
      0.50000000000000   0.50000000000000  -0.50000000000000  -0.50000000000000
      0.50000000000000  -0.50000000000000  -0.50000000000000   0.50000000000000

3.  HI =

      0.70710678118655                  0   0.70710678118655                  0
                     0   0.70710678118655                  0   0.70710678118655
      0.70710678118655                  0  -0.70710678118655                  0
                     0   0.70710678118655                  0  -0.70710678118655

4.  G =

      0.70710678118655   0.70710678118655                  0                  0
                     0                  0   0.70710678118655  -0.70710678118655
                     0                  0   0.70710678118655   0.70710678118655
      0.70710678118655  -0.70710678118655                  0                  0
```

Figure 16. Deutsch’s quantum algorithm simulation, Step 4: Quantum gate assembling, results of calculations

Fig. 17 shows the general results of decision-making Deutsch’s QA for fourth different cases.

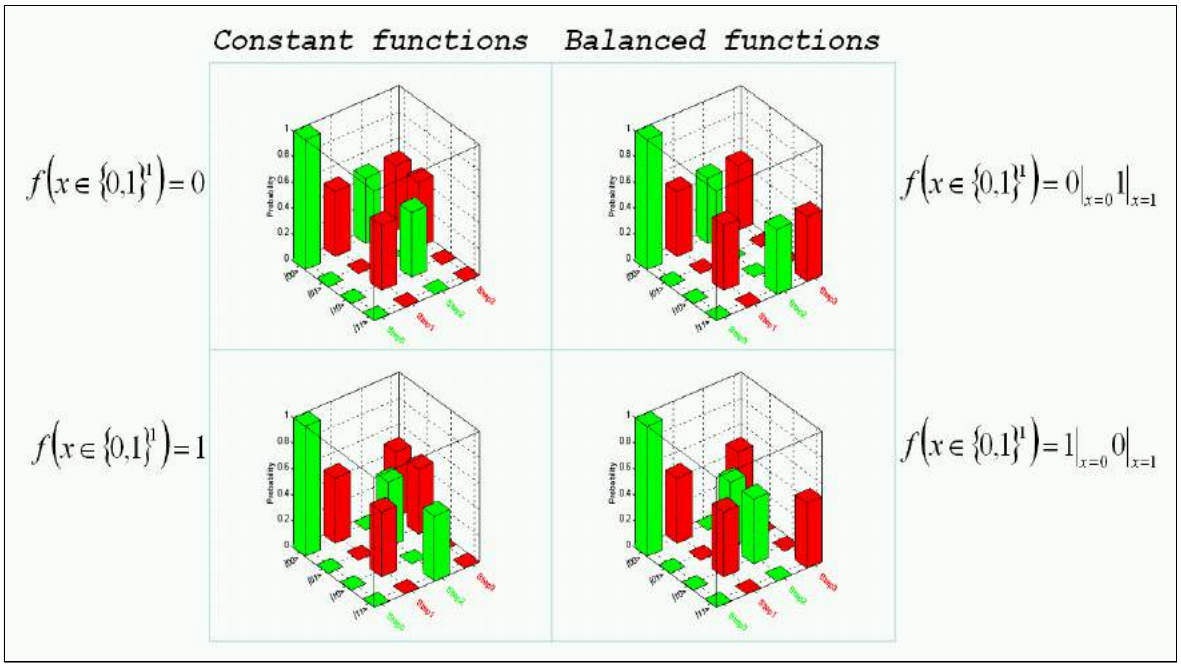


Figure 17. Deutsch quantum algorithm simulation: Algorithm 3d dynamics

Two-dimensional dynamic evolution of QAG for the case of constant function definition is shown in Fig. 18.

The case of balanced function definition the simulation result of QAG is shown in Fig. 19.

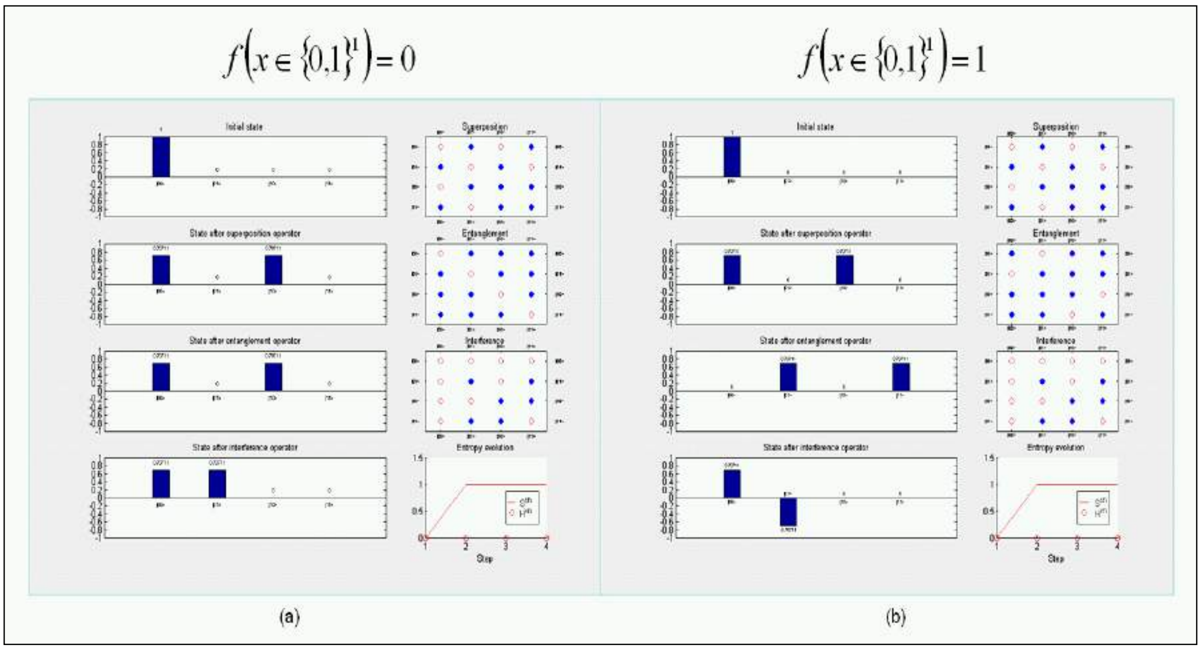


Figure 18. Deutsch quantum algorithm simulation: Algorithm 2d dynamics. Constant functions

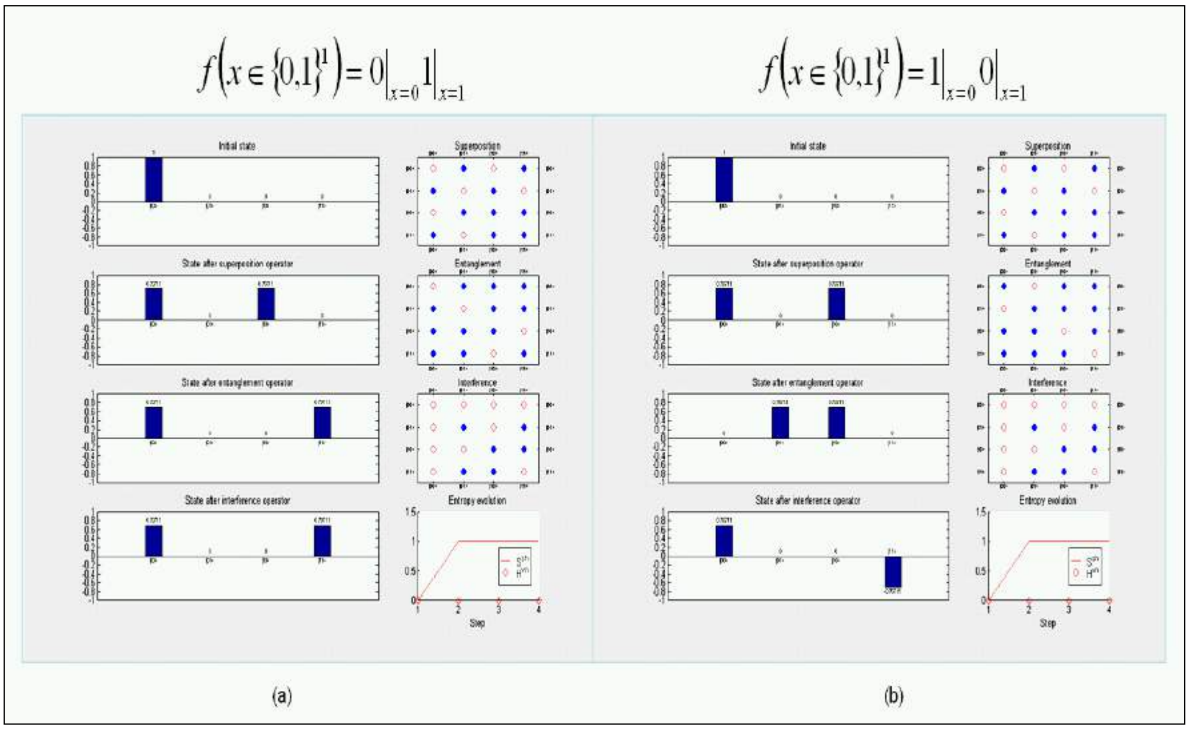
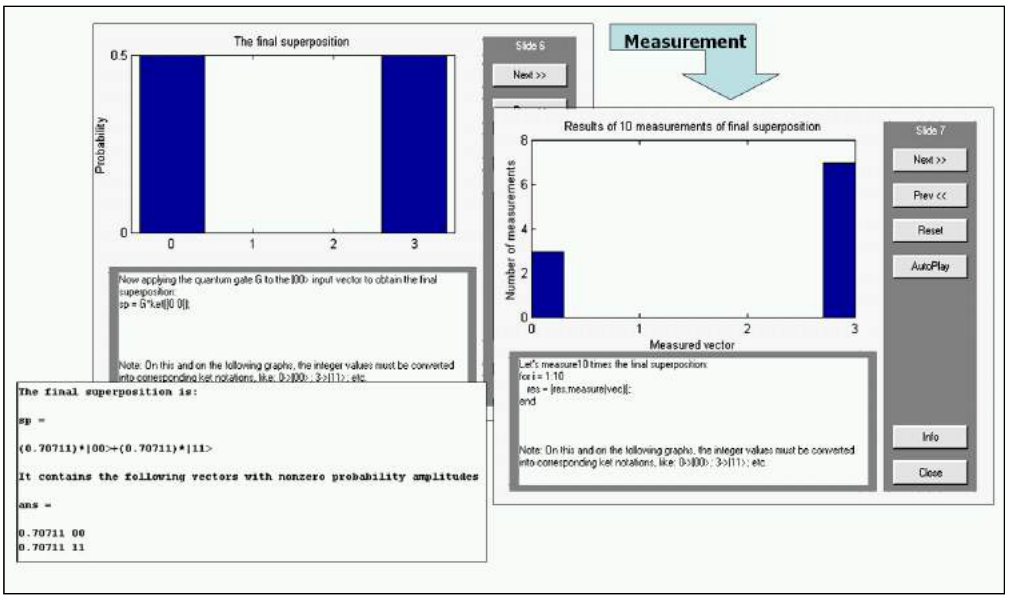


Figure 19. Deutsch quantum algorithm simulation: Algorithm 2d dynamics. Balanced functions

Figs 18 and 19 show also the entropy evaluation of the QAG for both cases.

These results are used for stopping criteria of the QA below.

Figs 20 and 21 show the results of final superposition measurement for definition of function property and its interpretation (decoding process), respectively.



Now applying the quantum gate G to the |00> input vector to obtain the final superposition:
sp = G*ket([0 0]);

Note: On this and on the following graphs, the integer values must be converted into corresponding ket notations, like: 0->|00> ; 3->|11> ; etc

```
The final superposition is:

sp =

(0.70711)*|00>+(0.70711)*|11>

It contains the following vectors with nonzero probability amplitudes

ans =

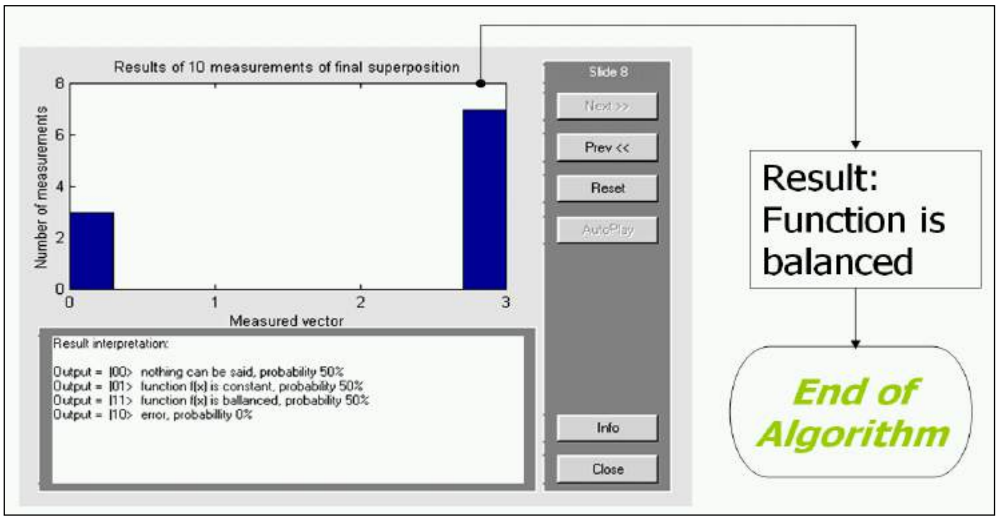
0.70711 00
0.70711 11
```

Let's measure10 times the final superposition:

```
for i = 1:10
  res = [res,measure(vec)];
end
```

Note: On this and on the following graphs, the integer values must be converted into corresponding ket notations, like: 0->|00> ; 3->|11> ; etc

Figure 20. Deutsch's quantum algorithm simulation, Step 5: Applying gate G to the input vector |00> and measurement



Result interpretation:

Output = |00> nothing can be said, probability 50%
Output = |01> function f(x) is constant, probability 50%
Output = |11> function f(x) is ballanced, probability 50%
Output = |10> error, probabillity 0%

Figure 21. Deutsch's quantum algorithm simulation, Step 6: Interpretation of results (decoding)

Deutsch-Jozsa's algorithm

The Deutsch-Jozsa's algorithm is based on the special form of its QAG. This example shows the importance of the structure of the matrix operator U_F.

Deutsch-Jozsa's problem. Definition of Deutsch-Jozsa's problem is stated as:

Input	*A constant or balanced function* $f:\{0,1\}^n \to \{0,1\}$
Problem	*Decide if f is constant or balanced*

This problem is very similar to Deutsch's problem, but it has been generalized to $n > 1$.

Fig. 22 shows the structure of the Problem and Fig. 23 shows the steps of gate design process.

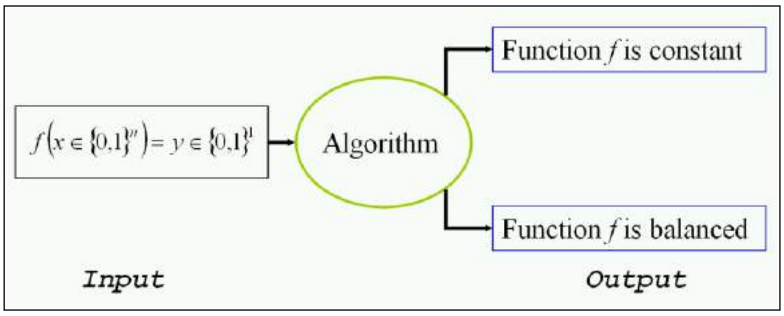


Figure 22. Deutsch-Jozsa's QA: Problem definition

According to design steps on the Fig. 23 consider Step 0: the Encoder.

Step 0

Encoder. As a threshold matter, it is useful to deal with some special functions with $n = 2$ to illuminate various aspects of this algorithm. Then the general case with $n = 2$ is discussed, and finally a balanced or constant function is encoded in the more general situation $n > 0$.

N	Definition of design step
0	Step 0: Encoder Step 0.1: Injective function *F* building Step 0.2: Preparation of map table for entanglement operator U_f

1	Step 1: Preparation of quantum operators Step 1.1: Preparation of superposition operator Step 1.2: Preparation of entanglement operator using information from step 0.2 Step 1.3: Preparation of interference operator Step 1.4: Quantum gate assembly
2	Step 2: Algorithm execution Step 2.1: Application of superposition operator Step 2.2: Application of entanglement operator Step 2.3: Application of interference operator Step 2.4: Measurement and interpretation of the output

Figure 23. Deutsch-Jozsa's QA: Steps of the algorithm design

Consider the encoding steps process according to the structure in the Fig. 6.

A. Encoding a constant function with value 1. Consider the case:

$$n = 2, \quad \forall x \in \{0,1\}^n : f(x) = 1 .$$

In this case, *f* map table is so defined:

x	***f*(*x*)**
00	1
01	1
10	1
11	1

The encoder block takes *f* map table as input and encodes it into matrix operator U_F, which acts inside of a complex Hilbert space.

Step 1

Function *f* is encoded into the injective function *F*, built according to the following statement:

$$F : \{0,1\}^{n+1} \to \{0,1\}^{n+1} : F(x_0, x_1, y_0) = (x_0, x_1, f(x_0, x_1) \oplus y_0)$$

Then *F* map table is:

(x_0, x_1, y_0)	**$F(x_0, x_1, y_0)$**
000	001
010	011
100	101
110	111
001	000
011	010
101	100
111	110

Step 2

Now encode *F* into U_F map table using the rule:

$$\forall t \in \{0,1\}^{n+1} : U_F\,[\tau(t)] = \tau[F(t)],$$

where τ is the code map defined above. This means:

|*x*₀ *x*₁ *y*₀>	***U*_F |*x*₀ *x*₁ *y*₀>**
|000>	|001>
|010>	|011>
|100>	|101>
|110>	|111>
|001>	|000>
|011>	|010>
|101>	|100>
|111>	|110>

Here, ket notation is used to denote basis vectors.

Step 3

Starting from the map table of U_F, calculate the corresponding matrix operator. This matrix is obtained using the rule:

$$[U_F]_{ij} = 1 \Leftrightarrow U_F \,|\, j\rangle = |i\rangle .$$

So, U_F is the following matrix:

U_F	|000>	|001>	|010>	|011>	|100>	|101>	|110>	|111>
|000>	0	1	0	0	0	0	0	0
|001>	1	0	0	0	0	0	0	0
|010>	0	0	0	1	0	0	0	0
|011>	0	0	1	0	0	0	0	0
|100>	0	0	0	0	0	1	0	0
|101>	0	0	0	0	1	0	0	0
|110>	0	0	0	0	0	0	0	1
|111>	0	0	0	0	0	0	1	0

Using matrix tensor product, U_F can be written as:

$$U_F = I \otimes I \otimes C = {}^2I \otimes C$$

where $\otimes$ is the tensor product, I is the identity matrix of order 2 and C is the NOT-matrix defined as: $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Matrix C flips a basis vector: in fact it transforms vector |0> into |1> and |1> into |0>.

If matrix U_F is applied to the tensor product of three vectors of dimension 2, the resulting vector is the tensor product of the three vectors obtained applying matrix I to the first two input vectors and matrix C to the third.

Tensor product and entanglemen. Given m vectors $\underline{v}_1$,.., $\underline{v}_m$ of dimension 2^{d_1},.., 2^{d_m} and m matrix operators M_1,.., M_m of order $2^{d_1}\times 2^{d_1}$,.., $2^{d_m}\times 2^{d_m}$ the following property holds:

$$\left(M_1 \otimes .. \otimes M_m\right) \cdot \left(\underline{v}_1 \otimes .. \otimes \underline{v}_n\right) = M_1 \cdot \underline{v}_1 \otimes ... \otimes M_m \cdot \underline{v}_n .$$

This means that, if a matrix operator can be written as the tensor product of *m* smaller matrix operator, the evolutions of the *m* vectors the operator is applied to are independent, namely no correlation is present among this vector. An important corollary is that if the initial state was not entangled, also the final state is not entangled. If, for example, $U_F = I \otimes I \otimes C$ then the structure of U_F is such that first two vectors in the input tensor product are preserved (action of *I*), whereas the third is flipped (action of *C*). One can easily verify that this action corresponds to the constraints stated by U_F map table.

B. *Encoding a constant function with value 0*. Now consider the case:

$$n = 2$$

$$\forall x \in \{0,1\}^n : f(x) = 0$$

In this case *f* map table is defined as:

x	***f*(*x*)**
00	0
01	0
10	0
11	0

Step 1

F map table is:

(x_0, x_1, y_0)	***F*(x_0, x_1, y_0)**
000	000
010	010
100	100
110	110
001	001
011	011
101	101
111	111

Step 2

F map table is encoded into U_F map table:

|x_0 x_1 y_0>	**U_F |x_0 x_1 y_0>**
|000>	|000>
|010>	|010>
|100>	|100>
|110>	|110>
|001>	|001>
|011>	|011>
|101>	|101>
|111>	|111>

Step 3

It is relatively easy to transform this map table into a matrix. Each vector is preserved.

Therefore, the corresponding matrix is the identity matrix of order 2^3.

U_F	|000>	|001>	|010>	|011>	|100>	|101>	|110>	|111>
|000>	1	0	0	0	0	0	0	0
|001>	0	1	0	0	0	0	0	0
|010>	0	0	1	0	0	0	0	0
|011>	0	0	0	1	0	0	0	0
|100>	0	0	0	0	1	0	0	0
|101>	0	0	0	0	0	1	0	0
|110>	0	0	0	0	0	0	1	0
|111>	0	0	0	0	0	0	0	1

Using matrix tensor product, this matrix can be written as:

$$U_F = I \otimes I \otimes I = {}^2I \otimes I.$$

The structure of U_F is such that all basis vectors of dimension 2 in the input tensor product evolve independently. No vector controls any other vector.

C. *Encoding a balanced function.* Consider now the balanced function:

$$n = 2, \ \ \forall (x_1, \dots, x_n) \in \{0,1\}^n : f(x_1, \dots, x_n) = x_1 \oplus \dots \oplus x_n.$$

In this case *f* map table is the following:

x	***f*(*x*)**
00	0
01	1
10	1
11	0

Step 1

The following map table calculated in the usual way represents the injective function *F* (where *f* is encoded into):

(x_0, x_1, y_0)	***F*(x_0, x_1, y_0)**
000	000
010	011
100	101
110	110

(x_0, x_1, y_0)	***F*(x_0, x_1, y_0)**
001	001
011	010
101	100
111	111

Step 2

Now encode F into U_F map table:

$\lvert x_0\, x_1\, y_0>$	$U_F \lvert x_0\, x_1\, y_0>$
|000>	|000>
|010>	|011>
|100>	|101>
|110>	|110>
|001>	|001>
|011>	|010>
|101>	|100>
|111>	|111>

Step 3

The matrix corresponding to U_F is:

U_F	|000>	|001>	|010>	|011>	|100>	|101>	|110>	|111>
|000>	1	0	0	0	0	0	0	0
|001>	0	1	0	0	0	0	0	0
|010>	0	0	0	1	0	0	0	0
|011>	0	0	1	0	0	0	0	0
|100>	0	0	0	0	0	1	0	0
|101>	0	0	0	0	1	0	0	0
|110>	0	0	0	0	0	0	1	0
|111>	0	0	0	0	0	0	0	1

This matrix cannot be written as the tensor product of smaller matrices.

It can be written as a block matrix as follows:

U_F	|00>	|01>	|10>	|11>
|00>	I	0	0	0
|01>	0	C	0	0
|10>	0	0	C	0
|11>	0	0	0	I

This means that the matrix operator acting on the third vector in the input tensor product depends on the values of the first two vectors. If these vectors are |0> and |0>, for instance, the operator acting on the third vector is the identity matrix, if the first two vectors are |0> and |1> then the evolution of the third is determined by matrix C.

This operator creates *entanglement,* namely correlation among the vectors in the tensor product. One cannot represent such an operator as a tensor product of simpler operators such as I and C in the same manner as it was possible in case of entanglement operators of constant functions presented above.

D. *General case with n* = 2. Consider now a general function with $n = 2$.

In this general case *f* map table is the following:

x	$f(x)$
00	f_{00}
01	f_{01}
10	f_{10}
11	f_{11}

with $f_i \in \{0,1\}$, i=00,01,10,11.

If f is constant then $\exists y \in \{0,1\} \forall x \in \{0,1\}^2 : f(x) = y$.

If f is balanced then $|\{f_i : f_i = 0\}|=|\{f_i : f_i = 1\}|$.

Step 1

Injective function F (where f is encoded) is represented by the following map table calculated in the usual way:

(x_0, x_1, y_0)	$F(x_0, x_1, y_0)$
000	$0\ 0\ f_{00}$
010	$0\ 1\ f_{01}$
100	$1\ 0\ f_{10}$
110	$1\ 1\ f_{11}$
001	$0\ 0\ \neg f_{00}$
011	$0\ 1\ \neg f_{01}$
101	$1\ 0\ \neg f_{10}$
111	$1\ 1\ \neg f_{11}$

Step 2

Now encode F into U_F map table:

$\|x_0\ x_1\ y_0>$	$U_F\ \|x_0\ x_1\ y_0>$
$\|000>$	$\|0\ 0\ f_{00}>$
$\|010>$	$\|0\ 1\ f_{01}>$
$\|100>$	$\|1\ 0\ f_{10}>$
$\|110>$	$\|1\ 1\ f_{11}>$
$\|001>$	$\|0\ 0\ \neg f_{00}>$
$\|011>$	$\|0\ 1\ \neg f_{01}>$
$\|101>$	$\|1\ 0\ \neg f_{10}>$
$\|111>$	$\|1\ 1\ \neg f_{11}>$

Step 3

The matrix corresponding to U_F can be written as a block matrix with the following general form:

$$\begin{array}{c|cccc}
U_F & |00> & |01> & |10> & |11> \\
\hline
|00> & \boxed{M_{00}} & 0 & 0 & 0 \\
|01> & 0 & \boxed{M_{01}} & 0 & 0 \\
|10> & 0 & 0 & \boxed{M_{10}} & 0 \\
|11> & 0 & 0 & 0 & \boxed{M_{11}}
\end{array}$$

where $M_i = I$ if $f_i = 0$ and $M_i = C$ if $f_i = 1$, i = 00,01,10,11.

The structure of this matrix is such that, when the first two vectors are mapped into some other vectors, the null operator is applied to the third vector, generating a null probability amplitude for this transition. This means that the first two vectors are always left unchanged. On the contrary, operators $M_i \in \{I, C\}$ and they are applied to the third vector when the first two are mapped into themselves. If all M_i coincide, operator U_F encodes a constant function.

Otherwise, it encodes a non-constant function.

If $|\{M_i: M_i = I\}|=|\{M_i: M_i = C\}|$ then f is balanced.

E. *General case.* Consider now the general case $n>0$. Input function f map table is the following:

$x\in\{0,1\}^n$	$f(x)$
0..0	$f_{0..0}$
0..1	$f_{0..1}$
…	...
1..1	$f_{1..1}$

with $f_i\in\{0,1\}$, $i\in\{0,1\}^n$.

If f is constant then $\exists y\in\{0,1\}\forall x\in\{0,1\}^n : f(x) = y$.

If f is balanced then $|\{f_i: f_i = 0\}| = |\{f_i: f_i = 1\}|$.

Step 1

The map table of the corresponding injective function F is:

$x\in\{0,1\}^{n+1}$	$F(x)$
0..00	0..0 $f_{0..0}$
…	…
1..10	1..1 $f_{1..1}$
0..01	0..0 $\neg f_{0..0}$
…	…
1..11	1..1 $\neg f_{1..1}$

Step 2

Now encode F into U_F map table:

$\|x>$	$U_F\|x>$
$\|0..00>$	$\|0..0\ f_{0..0}>$
…	…
$\|1..10>$	$\|1..1\ f_{1..1}>$
$\|0..01>$	$\|0..0\ \neg f_{0..0}>$
…	…
$\|1..11>$	$\|1..1\ \neg f_{1..1}>$

Step 3

The matrix corresponding to U_F can be written as a block matrix with the following general form:

$$\begin{array}{c|cccc} U_F & |0..0> & |0..1> & \dots & |1..1> \\ \hline |0..0> & \boxed{M_{0..0}} & 0 & 0 & 0 \\ |0..1> & 0 & \boxed{M_{0..1}} & 0 & 0 \\ \dots & \dots & \dots & \boxed{\dots} & \dots \\ |1..1> & 0 & 0 & 0 & \boxed{M_{1..1}} \end{array}$$

where $M_i = I$ if $f_i = 0$ and $M_i = C$ if $f_i = 1$, $i \in \{0,1\}^n$.

This matrix leaves the first n vectors unchanged and applies operator $M_i \in \{I, C\}$ to the last vector. If all M_i coincide with I or C, the matrix encodes a constant function and it can be written as ${}^nI \otimes I$ or ${}^nI \otimes C$. In this case *no entanglement* is generated.

Otherwise, if the condition $|\{M_i: M_i = I\}|=|\{M_i: M_i = C\}|$ is fulfilled, then f is balanced and the operator creates quantum correlation among vectors. It means that Deutsch-Jozsa's QA *needs bound amount of entanglement* and can be efficiently simulated on classical computer.

Matrix tensor and dot powers. Given a matrix M denote its k^{th}-power tensor product as: ${}^kM = M \otimes ... \otimes M \quad (k \ \ times)$. By contrast the k^{th}-power dot product is: $M^k = M \cdot ... \cdot M \quad (k \ \ times)$

Quantum block. Matrix U_F, the output of the encoder, is now embedded into the QAG of Deutsch-Jozsa's algorithm. As with Deutsch's algorithm, this gate is described using a quantum circuit in Fig. 24 (a).

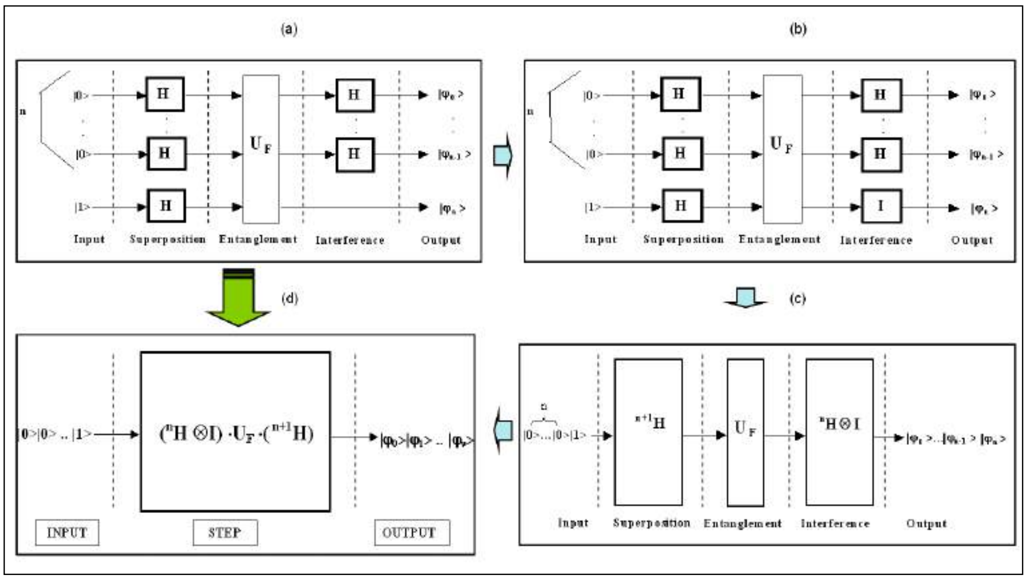


Figure 24. Deutsch-Jozsa's quantum algorithm simulation: Circuit representation and corresponding gate design

Using Rule 3 (see Fig. 7), similar to the case of Deutsch's QAG, compile the previous circuit into the one presented on the Fig. 24.

Now, consider operator U_F in the case of *constant* and *balanced* functions. The structure of this operator strongly influences the structure of the whole gate. It is possible to analyze this structure in the case f is 1 everywhere, f is 0 everywhere and in the general case with $n = 2$.

The general form for the gate with $n>0$ is given below.

A. *Constant function with value.* 1. If f is constant and its value is 1, matrix operator U_F can be written as ${}^nI \otimes$ C. This means, as it is stated by Rule 1 in Fig. 7, that U_F can be decomposed into n +1 smaller operators acting concurrently on the n+1 vectors of dimension 2 in the input tensor product.

The resulting circuit representation is shown according to Fig. 24 in Fig. 25.

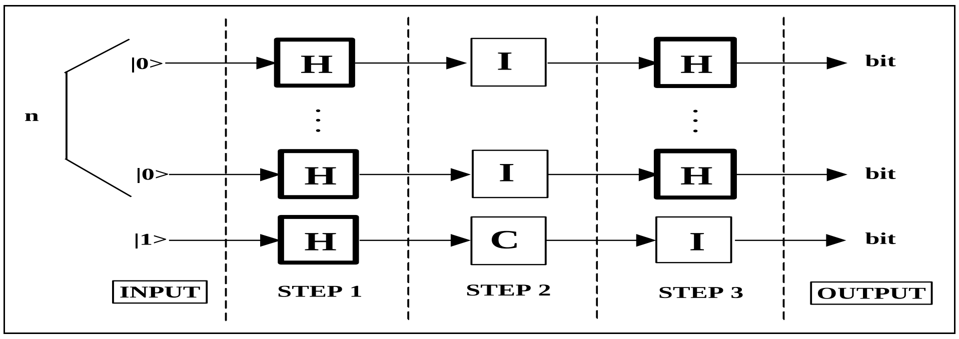


Figure 25. Constant Function with Value 1 — First Circuit

Now use Rule number 2 from Fig. 7 and find the sub-gate acting on every vector of dimension 2 in input. The result of this operation is shown in Fig. 26.

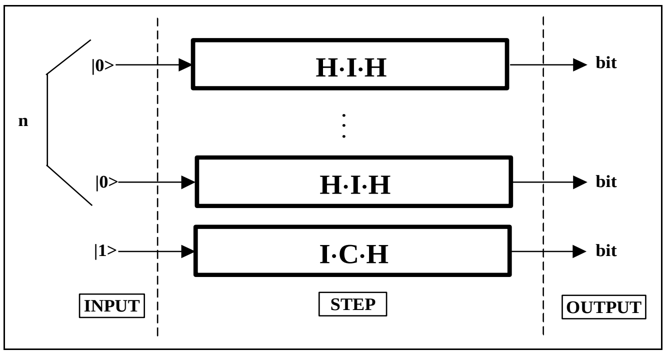


Figure 26. Constant Function with Value 1 — Second Circuit

Observe that every vector in input evolves independently from other vectors. This is because operator U_F doesn't create any correlation. So, the evolution of every input vector can be analyzed separately.

This circuit can be written in a simpler way as shown in Fig. 27, observing that $M \cdot I = M$.

It can be show that $H^2 = I$.

Therefore the circuit is rewritten in this way as shown in Fig. 28.

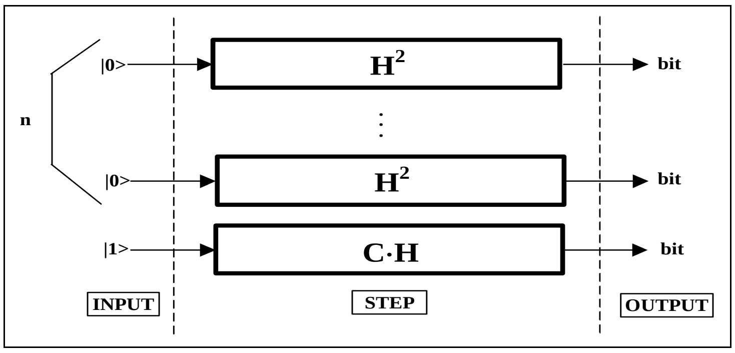


Figure 27. Constant Function with Value 1 — Third Circuit

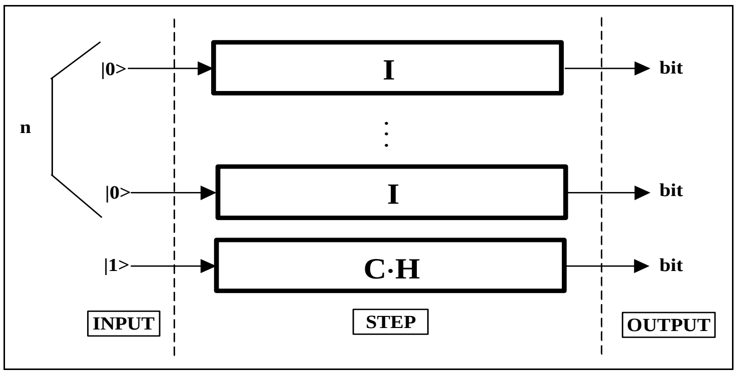


Figure 28. Constant Function with Value 1 — Fourth Circuit

Consider now the effect of the operators acting on every vector:

$$I\left|0\right\rangle=\left|0\right\rangle,\ C\cdot H\left|1\right\rangle=-\frac{\left|0\right\rangle-\left|1\right\rangle}{\sqrt{2}}.$$

Using these results in rule number 4 of Fig. 7 and applying Rule number 3 of Fig. 7, yields following circuit representation as shown on the Fig. 29 as the particular case of the structure shown in Fig. 24.

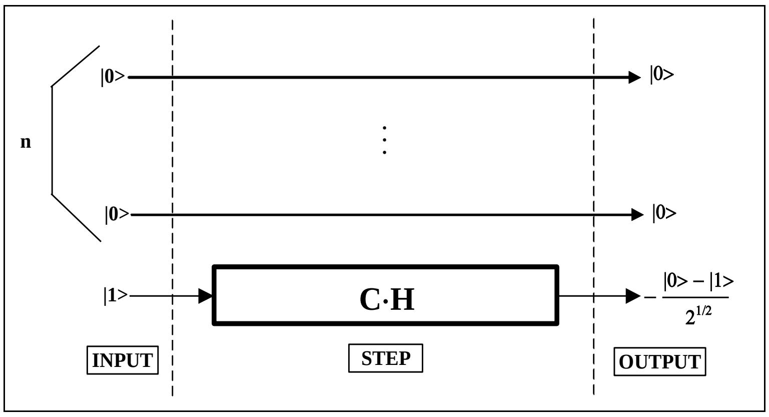


Figure 29. Constant Function with Value 1 — Fifth Circuit

It is easy to see that, if *f* is constant with value 1, the first *n* vectors are preserved.

B. *Constant function with value.* 0. A similar analysis can be repeated for a constant function with value 0. In this situation U_F can be written as $^nI \otimes I$ and the final circuit is shown on the Fig. 3.30. Also in this case, the first *n* input vectors are preserved. So, their output values after the QAG has acted are still |0>.

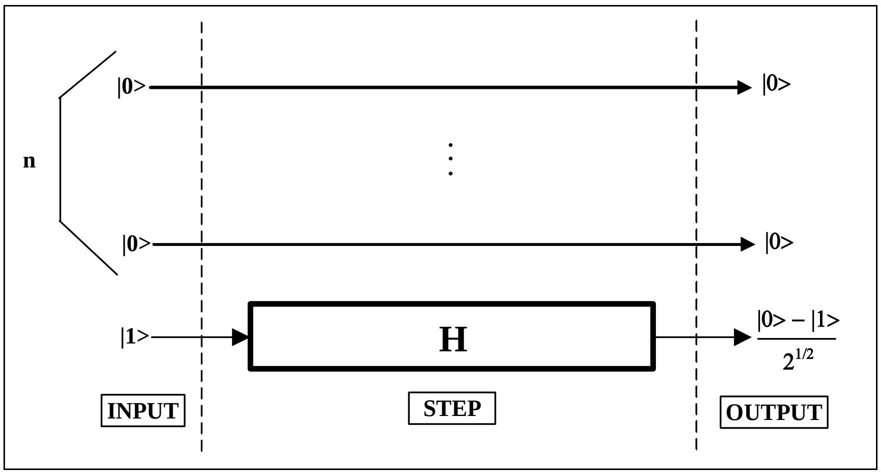


Figure 30. Constant Function with Value 0 — Final Circuit

C. *General case* (n = 2). The gate implementing Deutsch-Jozsa's algorithm in general case is obtained operating on the circuit of Figs 24c and 24d, with Rules 1 and 2 defined in Fig. 7. This is the circuit evolution as shown on the Figs 31and 32.

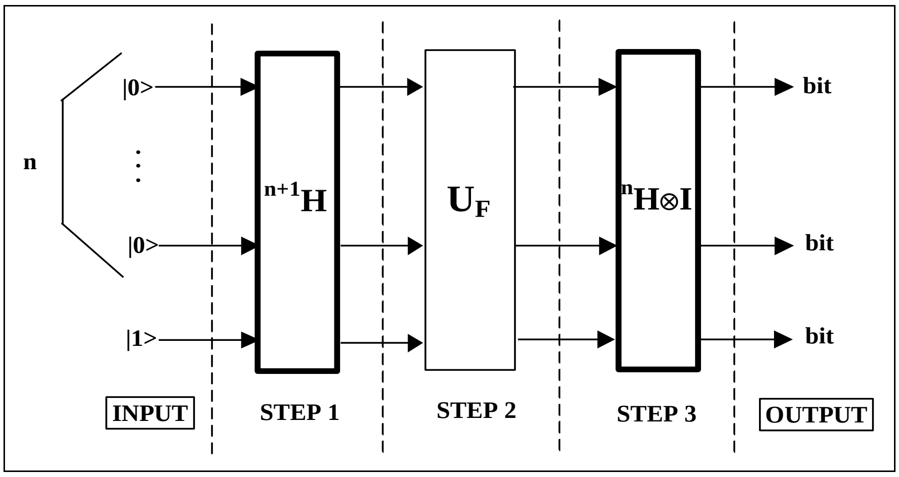


Figure 31. Evolution of the circuit in Fig. 24 (c)

If n = 2, U_F has the following form:

U_F	|00>	|01>	|10>	|11>
|00>	M_{00}	0	0	0
|01>	0	M_{01}	0	0
|10>	0	0	M_{10}	0
|11>	0	0	0	M_{11}

where $M_i \in \{I, C\}$, i = 00,01,10,11.

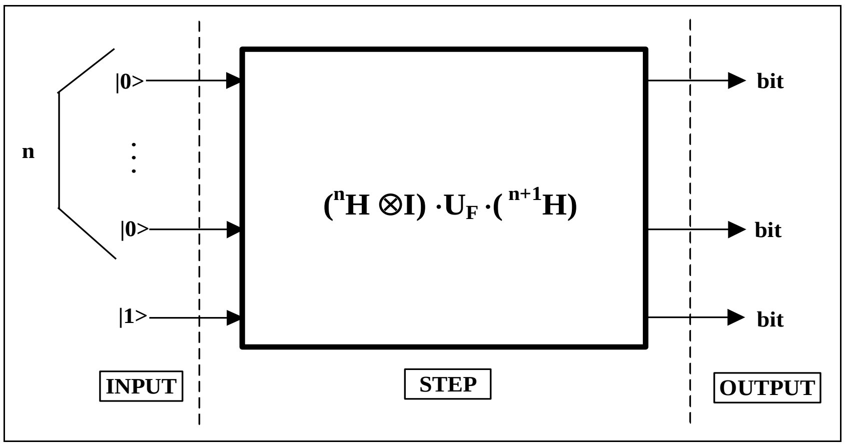


Figure 32. Deutsch-Jozsa's quantum gate

Calculate the QG $G=(^2H \otimes I) \cdot U_F \cdot (^{2+1}H)$ in this case:

^3H	|00>	|01>	|10>	|11>
|00>	$H/2$	$H/2$	$H/2$	$H/2$
|01>	$H/2$	$-H/2$	$H/2$	$-H/2$
|10>	$H/2$	$H/2$	$-H/2$	$-H/2$
|11>	$H/2$	$-H/2$	$-H/2$	$H/2$

${}^2H\otimes I$	|00>	|01>	|10>	|11>
|00>	$I/2$	$I/2$	$I/2$	$I/2$
|01>	$I/2$	$-I/2$	$I/2$	$-I/2$
|10>	$I/2$	$I/2$	$-I/2$	$-I/2$
|11>	$I/2$	$-I/2$	$-I/2$	$I/2$

$U_F \cdot {}^3H$	|00>	|01>	|10>	|11>
|00>	$M_{00}H/2$	$M_{00}H/2$	$M_{00}H/2$	$M_{00}H/2$
|01>	$M_{01}H/2$	$-M_{01}H/2$	$M_{01}H/2$	$-M_{01}H/2$
|10>	$M_{10}H/2$	$M_{10}H/2$	$- M_{10}H/2$	$- M_{10}H/2$
|11>	$M_{11}H/2$	$- M_{11}H/2$	$- M_{11}H/2$	$M_{11}H/2$

G	|00>	|01>	|10>	|11>
|00>	$(M_{00}+M_{01}+M_{10}+M_{11})H/4$	$(M_{00}-M_{01}+M_{10}-M_{11})H/4$	$(M_{00}+M_{01}-M_{10}-M_{11})H/4$	$(M_{00}-M_{01}-M_{10}+M_{11})H/4$
|01>	$(M_{00}-M_{01}+M_{10}-M_{11})H/4$	$(M_{00}+M_{01}+M_{10}+M_{11})H/4$	$(M_{00}-M_{01}-M_{10}+M_{11})H/4$	$(M_{00}+M_{01}-M_{10}-M_{11})H/4$
|10>	$(M_{00}+M_{01}-M_{10}-M_{11})H/4$	$(M_{00}-M_{01}-M_{10}+M_{11})H/4$	$(M_{00}+M_{01}+M_{10}+M_{11})H/4$	$(M_{00}-M_{01}+M_{10}-M_{11})H/4$
|11>	$(M_{00}-M_{01}-M_{10}+M_{11})H/4$	$(M_{00}+M_{01}-M_{10}-M_{11})H/4$	$(M_{00}-M_{01}+M_{10}-M_{11})H/4$	$(M_{00}+M_{01}+M_{10}+M_{11})H/4$

Now, consider the application of G to vector |001>:

$$G|001\rangle = \frac{1}{4}|00\rangle \otimes (M_{00}+M_{01}+M_{10}+M_{11})H|1\rangle + \frac{1}{4}|01\rangle \otimes (M_{00}-M_{01}+M_{10}-M_{11})H|1\rangle +$$
$$+\frac{1}{4}|10\rangle \otimes (M_{00}+M_{01}-M_{10}-M_{11})H|1\rangle + \frac{1}{4}|11\rangle \otimes (M_{00}-M_{01}-M_{10}+M_{11})H|1\rangle$$

Consider the operator $(M_{00}+M_{01}+M_{10}+M_{11})H$ under the hypotheses of balanced functions $M_i\in\{I, C\}$and $|\{M_i\colon M_i = I\}| = |\{M_i\colon M_i = C\}|$. Then:

$M_{00}+M_{01}+M_{10}+M_{11}$	|0>	|1>
|0>	2	2
|1>	2	2

$(M_{00}+M_{01}+M_{10}+M_{11})H/4$	|0>	|1>
|0>	$1/2^{1/2}$	0
|1>	$1/2^{1/2}$	0

Thus:

$$\frac{1}{4}\left(M_{00}+M_{01}+M_{10}+M_{11}\right)H\left|1\right\rangle = 0\,.$$

This means that the probability amplitude of vector |001> of being mapped into a vector |000> or |001> is null.

Consider now the operators:

$$(M_{00}+M_{01}+M_{10}+M_{11})H$$

$$(M_{00}-M_{01}+M_{10}-M_{11})H$$

$$(M_{00}+M_{01}-M_{10}-M_{11})H$$

$$(M_{00}-M_{01}-M_{10}+M_{11})H$$

under the hypotheses $\forall i$: $M_{\mathrm{i}} = I$, which holds for constant functions with values 0:

$M_{00}+M_{01}+M_{10}+M_{11}$	$\vert 0>$	$\vert 1>$
$\vert 0>$	4	0
$\vert 1>$	0	4
$(M_{00}+M_{01}+M_{10}+M_{11})H/4$	$\vert 0>$	$\vert 1>$
$\vert 0>$	$1/2^{1/2}$	$1/2^{1/2}$
$\vert 1>$	$1/2^{1/2}$	$-1/2^{1/2}$

$M_{00}-M_{01}+M_{10}-M_{11}$	$\vert 0>$	$\vert 1>$
$\vert 0>$	0	0
$\vert 1>$	0	0
$M_{00}+M_{01}-M_{10}-M_{11}$	$\vert 0>$	$\vert 1>$
$\vert 0>$	0	0
$\vert 1>$	0	0
$M_{00}-M_{01}-M_{10}+M_{11}$	$\vert 0>$	$\vert 1>$
$\vert 0>$	0	0
$\vert 1>$	0	0

Using these calculations, the following results are obtained:

$$\frac{1}{4}\left(M_{00}-M_{01}+M_{10}-M_{11}\right)H\left|1\right\rangle = 0,$$

$$\frac{1}{4}\left(M_{00}+M_{01}-M_{10}-M_{11}\right)H\left|1\right\rangle = 0,$$

$$\frac{1}{4}\left(M_{00}-M_{01}-M_{10}+M_{11}\right)H\left|1\right\rangle = 0.$$

This means that the probability amplitude of vector |001> of being mapped into a superposition of vectors |010>, |011>, |100>, |101>, |110>, |111> is null. The only possible output is a superposition of vectors |000> and |001>, as shown before using circuits. A similar analysis can be developed under the hypotheses $\forall i$: $M_i = C$.

It is useful to outline the evolution of the probability amplitudes of every basis vector while operator ^3H, U_F and $^2H \otimes I$ are applied in sequence, for instance when f has constant value 1. This is shown in Fig. 33.

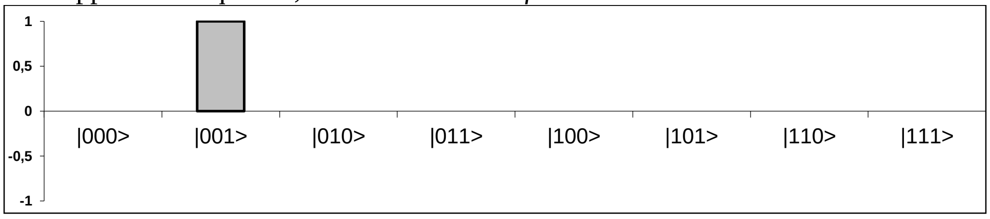


Figure 33 (a). Input probability amplitudes

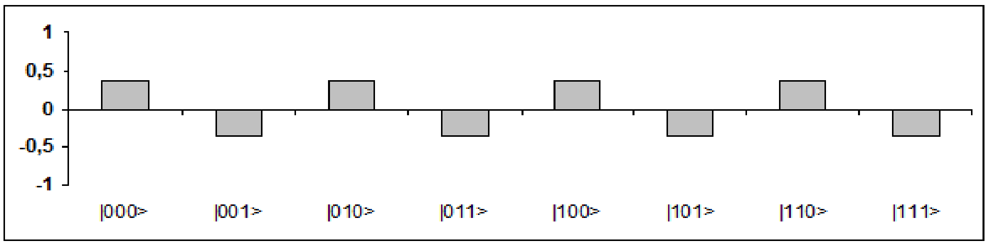


Figure 33 (b). Probability amplitudes after Step 1

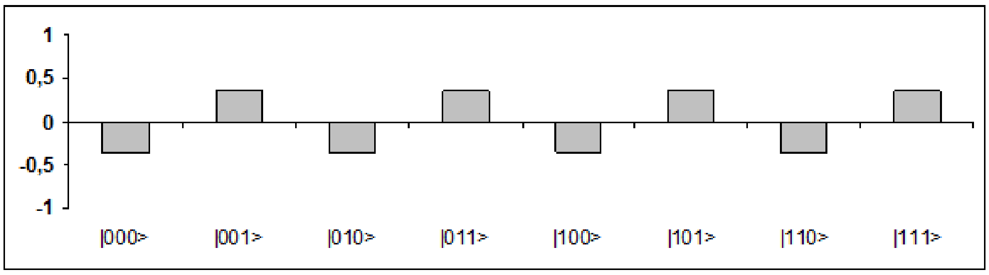


Figure 33 (c). Probability amplitudes after Step 2

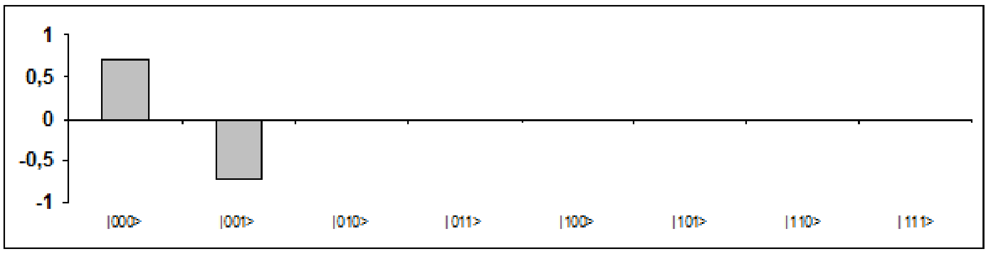


Figure 33 (d). Probability amplitudes after Step 3

Operator ^3H in Fig. 33 (b) puts the initial canonical basis vector |001> into a superposition of all basis vectors with the same (real) coefficients in modulus, but with positive sign if the last vector is |0>, negative otherwise.

Operator U_F in Fig. 33 (c) in this case doesn't create correlation: it flips the third vector independently from the values of the first two vectors.

Finally, $^2H \otimes I$ in Fig. 33 (d) produces constructive interference: for every basis vector $|\underline{x}_0\underline{x}_1\underline{y}_0>$ it calculates its output probability amplitude $\alpha'_{\underline{x}_0\underline{x}_1\underline{y}_0}$ as the summation of the probability amplitudes of all basis vectors in the form $|x_0x_1\underline{y}_0>$ in the input superposition, all with the same sign if $|\underline{x}_0\underline{x}_1> = |00>$, otherwise changing the sign of exactly the middle of the probability amplitudes. Since, in this case, the vectors in the form $|x_0x_10>$ have the same (negative real) probability amplitude and vectors in the form $|x_0x_11>$ have the same (positive real) probability amplitude, when $|\underline{x}_0\underline{x}_1> = |00>$, probability amplitudes interfere positively. Otherwise the terms in the summation interfere destructively annihilating the result.

D. *General case* ($n > 0$). In the general case $n > 0$, U_F has the following form:

U_F	$\|0..0>$	$\|0..1>$	…	$\|1..1>$
$\|0..0>$	$M_{0..0}$	0	0	0
$\|0..1>$	0	$M_{0..1}$	0	0
…	…	…	…	…
$\|1..1>$	0	0	0	$M_{1..1}$

where $M_i \in \{I, C\}$, $i \in \{0,1\}^n$.

Calculate the QAG $G=({}^nH \otimes I) \cdot U_F \cdot ({}^{n+1}H)$:

${}^{n+1}H$	$\|0..0>$	…	$\|j>$	…	$\|1..1>$
$\|0..0>$	$H/2^{n/2}$	…	$H/2^{n/2}$	…	$H/2^{n/2}$
…	…	…	…	…	…
$\|i>$	$H/2^{n/2}$	…	$(-1)^{i \cdot j}H/2^{n/2}$	…	$(-1)^{i \cdot (1..1)}H/2^{n/2}$
…	…	…	…	…	…
$\|1..1>$	$H/2^{n/2}$	…	$(-1)^{(1..1) \cdot j}H/2^{n/2}$	…	$(-1)^{(1..1) \cdot (1..1)}H/2^{n/2}$

Here the binary string operator, which represents the parity of the AND bit per bit between two strings, is used.

Priority of bit per bit AND Given two binary strings *x* and *y* of length *n*, define:

$$x \cdot y = x_1 \cdot y_1 \oplus x_2 \cdot y_2 \oplus ... \oplus x_n \cdot y_n$$

The symbol «·» used between two bits is interpreted as the logical AND operator.

It can be shown that the matrix ${}^{n+1}H$ really has the described form. It can be shown that:

$$\left[{}^nH\right]_{i\,j} = \frac{(-1)^{i \cdot j}}{2^{n/2}},$$

The proof is by induction: $n = 1$:

$$\left[{}^1H\right]_{0,0} = \frac{1}{2^{1/2}} = \frac{(-1)^{(0)\cdot(0)}}{2^{1/2}} \quad \left[{}^1H\right]_{0,1} = \frac{1}{2^{1/2}} = \frac{(-1)^{(0)\cdot(1)}}{2^{1/2}}$$

$$\left[{}^1H\right]_{1,0} = \frac{1}{2^{1/2}} = \frac{(-1)^{(1)\cdot(0)}}{2^{1/2}} \quad \left[{}^1H\right]_{1,1} = \frac{-1}{2^{1/2}} = \frac{(-1)^{(1)\cdot(1)}}{2^{1/2}}$$

$n > 1$:

$$\left[{}^{n}H\right]_{i0,j0}=\frac{1}{2^{1/2}}\left[{}^{n-1}H\right]_{i,j}=\frac{1}{2^{1/2}}\frac{(-1)^{i\cdot j}}{2^{(n-1)/2}}=\frac{(-1)^{(i0)\cdot(j0)}}{2^{n/2}}$$

$$\left[{}^{n}H\right]_{i0,j1}=\frac{1}{2^{1/2}}\left[{}^{n-1}H\right]_{i,j}=\frac{1}{2^{1/2}}\frac{(-1)^{i\cdot j}}{2^{(n-1)/2}}=\frac{(-1)^{(i0)\cdot(j1)}}{2^{n/2}}$$

$$\left[{}^{n}H\right]_{i1,j0}=\frac{1}{2^{1/2}}\left[{}^{n-1}H\right]_{i,j}=\frac{1}{2^{1/2}}\frac{(-1)^{i\cdot j}}{2^{(n-1)/2}}=\frac{(-1)^{(i1)\cdot(j0)}}{2^{n/2}}$$

$$\left[{}^{n}H\right]_{i1,j1}=-\frac{1}{2^{1/2}}\left[{}^{n-1}H\right]_{i,j}=-\frac{1}{2^{1/2}}\frac{(-1)^{i\cdot j}}{2^{(n-1)/2}}=\frac{(-1)^{(i1)\cdot(j1)}}{2^{n/2}}$$

Matrix ${}^{n+1}H$ is obtained from ${}^{n}H$ by tensor product. Similarly, matrix ${}^{n}H \otimes I$ is calculated:

${}^{n}H\otimes I$	|0..0>	…	|j>	…	|1..1>
|0..0>	$I/2^{n/2}$	…	$I/2^{n/2}$	…	$I/2^{n/2}$
…	…	…	…	…	…
|i>	$I/2^{n/2}$	…	$(-1)^{i\cdot j}I/2^{n/2}$	…	$(-1)^{i\cdot(1..1)}I/2^{n/2}$
…	…	…	…	…	…
|1..1>	$I/2^{n/2}$	…	$(-1)^{(1..1)\cdot j}I/2^{n/2}$	…	$(-1)^{(1..1)\cdot(1..1)}I/2^{n/2}$

$U_F\cdot{}^{n+1}H$	|0..0>	…	|j>	…	|1..1>
|0..0>	$M_{0..0}H/2^{n/2}$	…	$M_{0..0}H/2^{n/2}$	…	$M_{0..0}H/2^{n/2}$
…	…	…	…	…	…
|i>	$M_iH/2^{n/2}$	…	$(-1)^{i\cdot j}M_iH/2^{n/2}$	…	$(-1)^{i\cdot(1..1)}M_iH/2^{n/2}$
…	…	…	…	…	…
|1..1>	$M_{1..1}H/2^{n/2}$	…	$(-1)^{(1..1)\cdot j}M_{1..1}H/2^{n/2}$	…	$(-1)^{(1..1)\cdot(1..1)}M_{1..1}H/2^{n/2}$

Only the first column of gate G is calculated since this operator is applied exclusively to input vector |0…01> and so only the first column is involved.

G	|0..0>	…
|0...0>	$(M_{0..0}+..+M_i+..+M_{1..1})H/2^n$	…
…	…	…
|i>	$(\Sigma_{j\in\{0,1\}^n}(-1)^{i\cdot j}M_j)H/2^n$	…
…	…	…
|1...1>	$(\Sigma_{j\in\{0,1\}^n}(-1)^{(1..1)\cdot j}M_j)H/2^n$	…

Now consider the case of f constant. This means that all matrices M_i are identical.

This implies:

$$\frac{1}{2^n}\left(\sum_j(-1)^{i\cdot j}M_j\right)H=0,$$

In this summation the number of +1 equals the number of –1. Therefore, the input vector |0…01> is mapped into a superposition of vectors |0…00> and |0…01> as shown using circuits.

If f is balanced, the number of $M_i = I$ equals the number of $M_i = C$. This implies:

$$\frac{1}{2^n}\left(\sum_j M_j\right)H = \frac{1}{2^n}\left(2^{n-1}I + 2^{n-1}C\right)H = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}H =$$

$$= \frac{1}{2\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}.$$

And therefore:

$$\frac{1}{2^n}\left(\sum_j M_j\right)H\left|1\right\rangle = 0\,.$$

This means that input vector |0…01>, in the case of balanced functions, cannot be mapped by the QAG into a superposition containing vectors |0…00> or |0…01>.

The quantum block terminates with measurement. The above results show the possible outputs of measurement and their probabilities:

Superposition of Basis Vectors (*Before Measurement*)	Result of Measurement	
	Vector	*Probability*
Constant functions: G|0…01>=|0..0>⊗(α_0|0>+α_1|1>)	|0..00> |0..01>	$\|\|\alpha_0\|\|^2$ $\|\|\alpha_1\|\|^2$
Balanced functions: G|0...01>=$\Sigma_{i\in\{0,1\}^n - \{0..00,\, 0..01\}}\ \alpha_i$ |i>	$\forall i\in\{0,1\}^n$–{0..00, 0..01}:|i>	$\|\|\alpha_i\|\|^2$

The set A–B is given by all elements of A, unless those elements belonging to B too. This set is sometimes denoted as A/B. The quantum block is repeated only one time in Deutsch-Jozsa's algorithm. So, the final collection is made only by one vector.

Decoder. As in Deutsch's algorithm, when the final basis vector has been measured, one must interpret it in order to decide if f is constant or balanced. If the resulting vector is |0…0> it is known that the function was constant, otherwise it is balanced. In fact gate G produces a vector such that, when it is measured, only basis vectors |0…00> and |0…01> have a non-null probability amplitude exclusively in the case f is constant. Besides, if f is balanced, these two vectors have null coefficients in the linear combination of basis vectors generated by G. In this way, the resulting vector is decoded in order to answer Deutsch-Jozsa's problem:

Resulting Vector (*after measurement*)	Answer
|0…00>	f is constant
|0…01>	f is constant
otherwise	f is balanced

Computer design process of Deutsch-Jozsa QAG (D.–J. QAG) and simulation results. Consider the design process of the Deutsch-Josa QAG according to the steps represented in Fig. 23.

For *step* 0 (encoding), case $n = 3$, examples of constant and balanced functions encoding are shown in Figs 34 and 35.

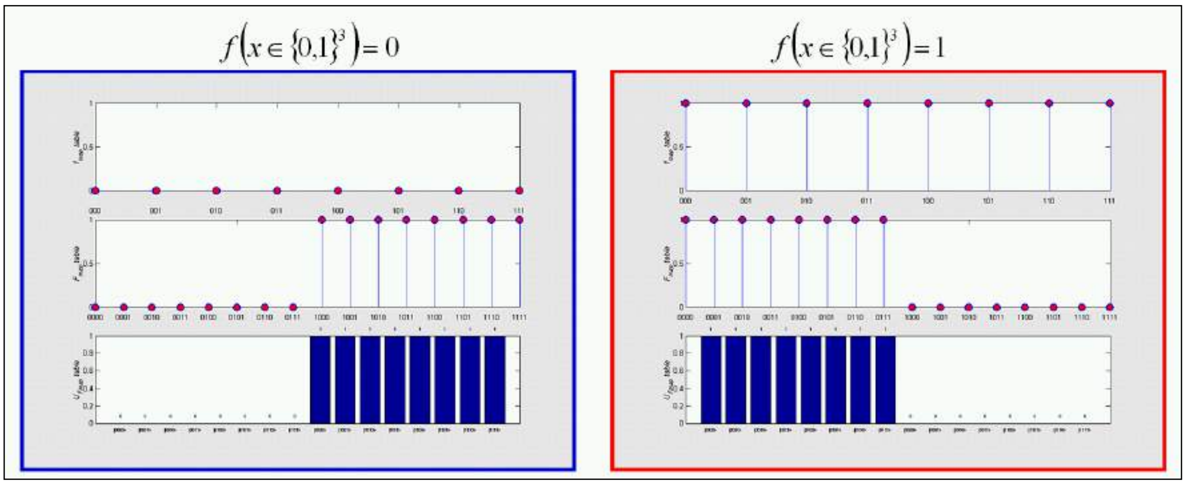


Figure 34. Deutsch-Jozsa's QA: Step 0. Constant functions encoding. Order n = 3

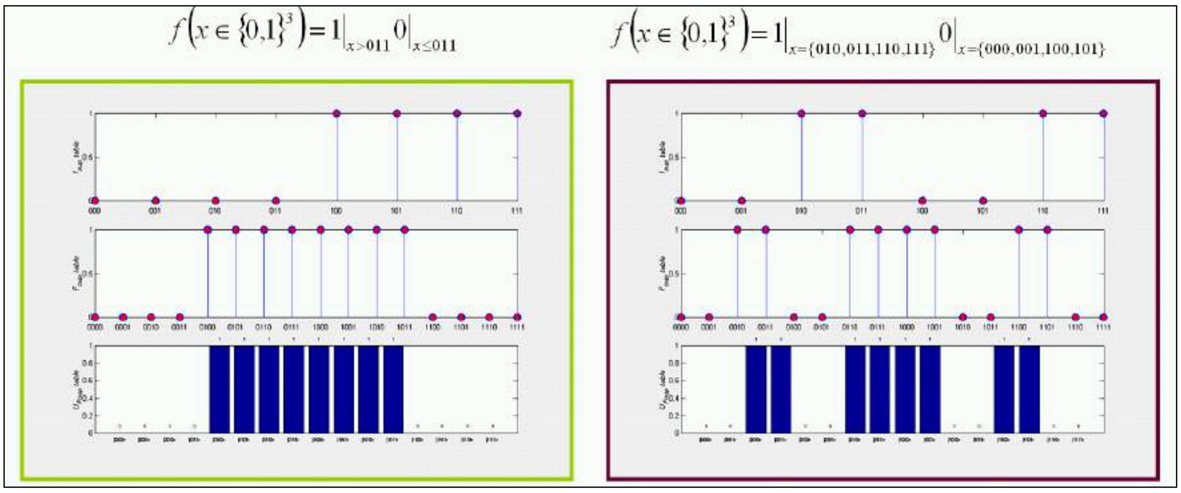
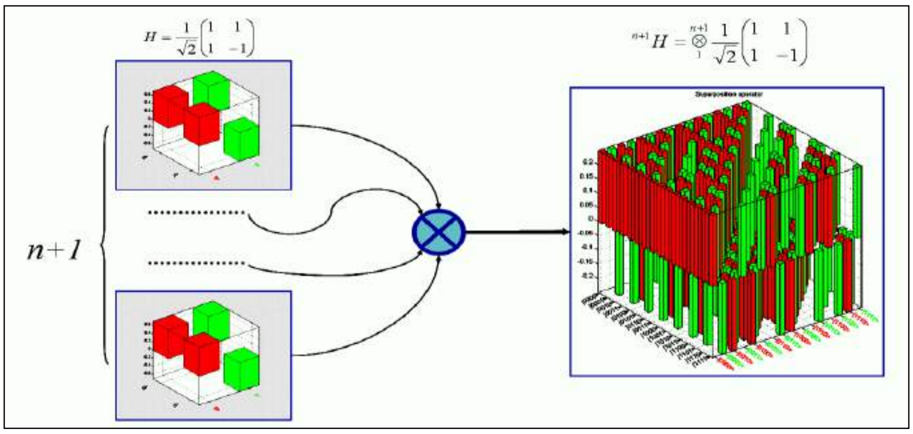


Figure 35. Deutsch-Jozsa's QA: Step 0. Balanced functions encoding. Order n = 3

For step 1 in Fig. 23, the example of quantum operator preparation such as superposition operator in Fig. 36 is shown.



$H=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$
$^{n+1}H=\bigotimes_{1}^{n+1}\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$
Superposition operator
$n+1$

Figure 36. Deutsch-Jozsa's QA: Step 1.1. Preparation of quantum operators: Superposition operator

Figs 37-40 shows the step 1.2 from Fig. 23 as the preparation of entanglement operators:

For a *constant function*:

$$f\left(\in\{0,1\}^3=0\right) \text{ and } f\left(\in\{0,1\}^3=1\right)$$

as shown in Figs 37 and 38;

For a *balanced function*:

$$f\left(\in\{0,1\}^3=1\big|_{x>011}0\big|_{x\le 011}\right) \text{ and } f\left(\in\{0,1\}^3=\begin{cases}1 & x=\{010,011,110,111\}\\ 0 & x=\{000,001,100,101\}\end{cases}\right),$$

as shown in Figs 39 and 40 respectively.

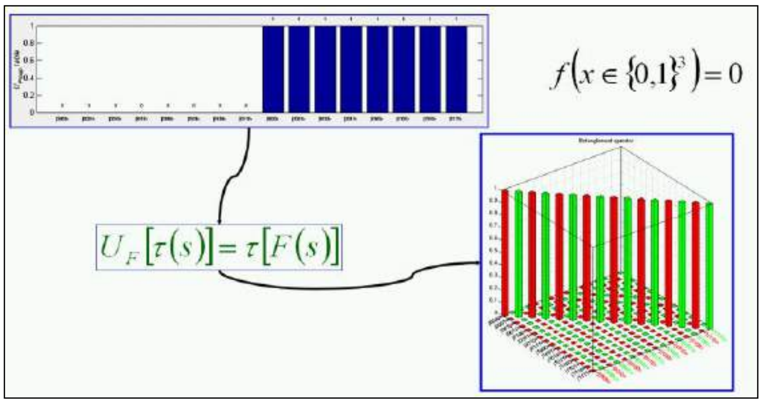


Figure 37. Deutsch-Jozsa's QA: Step 1.2. Preparation of quantum operators: Entanglement operator

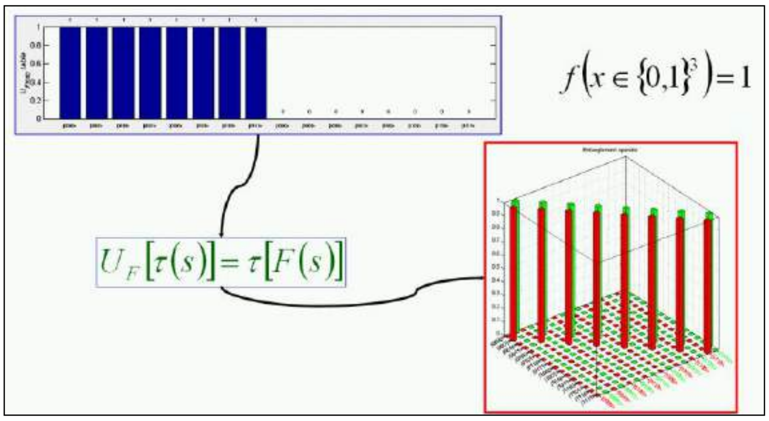


Figure 38. Deutsch-Jozsa's QA: Step 1.2. Preparation of quantum operators: Entanglement operator

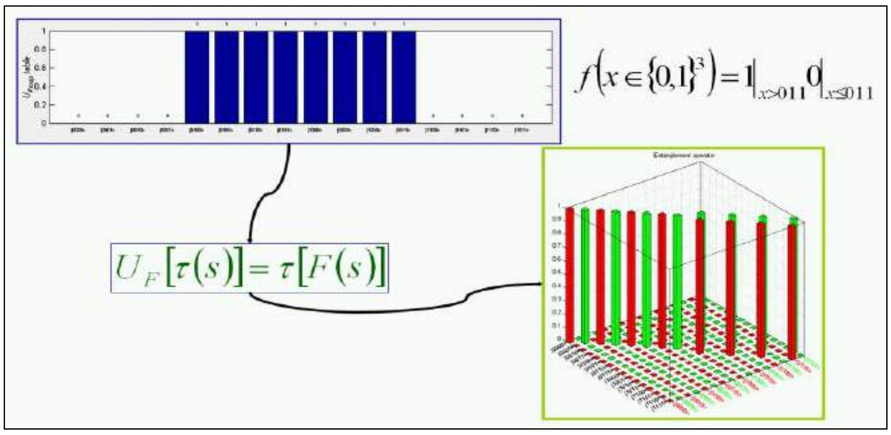

$f\left(x\in\{0,1\}^3\right)=1\big|_{x>011}0\big|_{x\le 011}$
$U_F[\tau(s)]=\tau[F(s)]$
Entanglement operator

Figure 39. Deutsch-Jozsa's QA: Step 1.2. Preparation of quantum operators: Entanglement operator

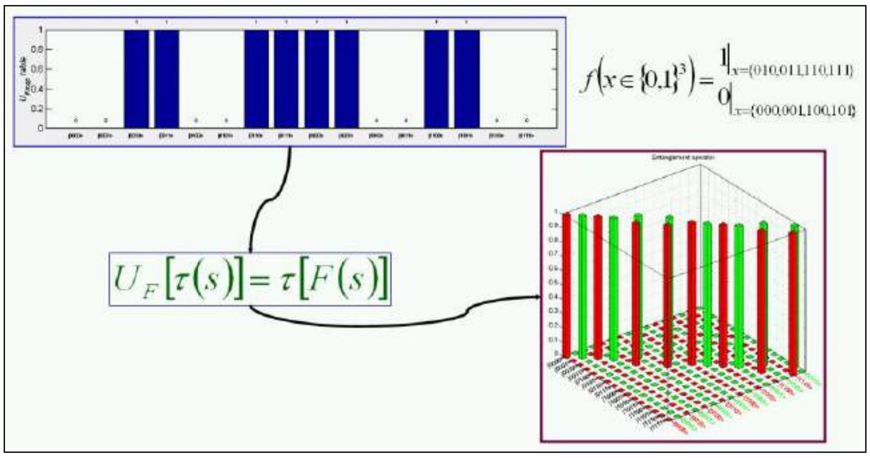


Figure 40. Deutsch-Jozsa's QA: Step 1.2. Preparation of quantum operators: Entanglement operator

Step 1.3 in Fig. 23 shows the preparation of the interference operator and is shown in Fig. 41. Comparison between superposition and interference operators is shown in Fig. 42.

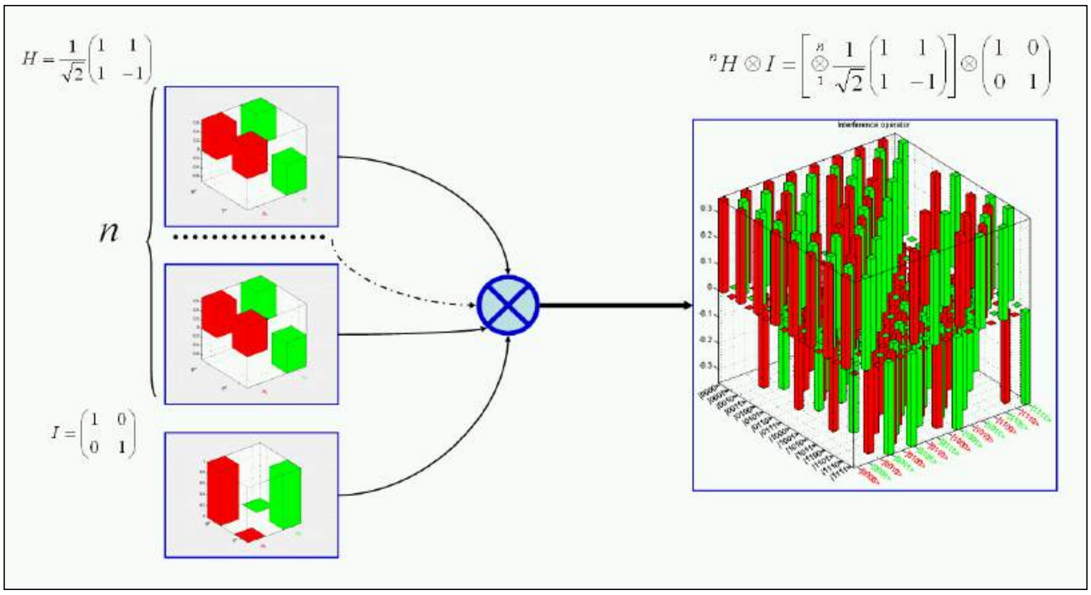

$H=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$
n
$I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$
${}^{n}H\otimes I=\left[\bigotimes_{1}^{n}\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\right]\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}$
Interference operator

Figure 41. Deutsch-Jozsa's QA: Step 1.3. Preparation of quantum operators: Interference operator

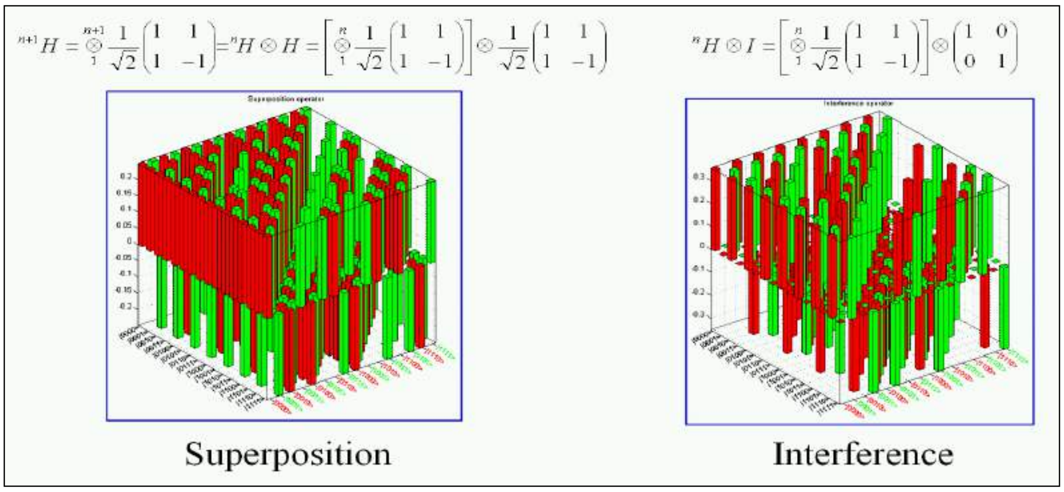


Figure 42. Deutsch-Jozsa's QA: Superposition and interference operators

The evolution of gate design process from Fig. 23 is shown in Fig. 43.

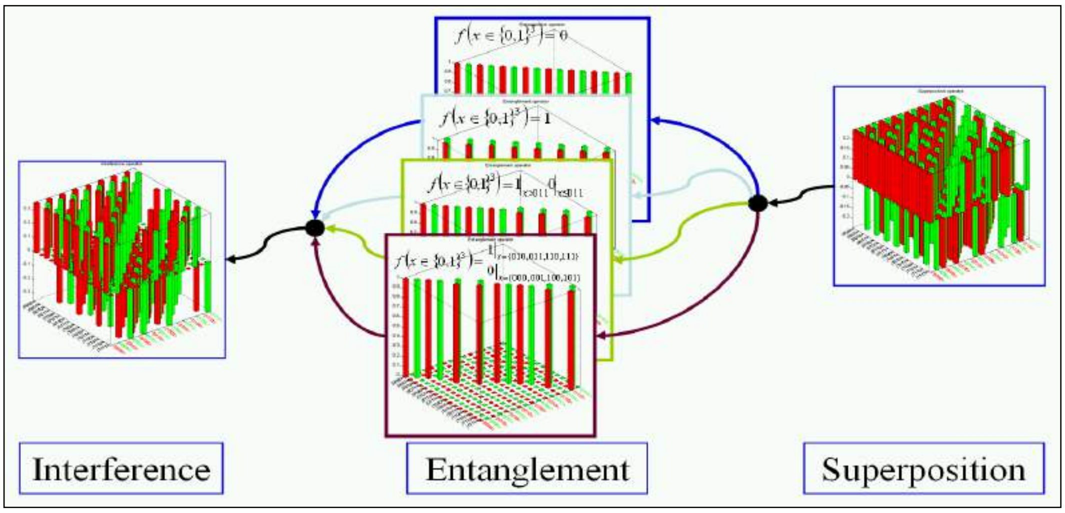

$f(x \in \{0,1\}^3) = 0$
$f(x \in \{0,1\}^3) = 1$
$f(x \in \{0,1\}^3) = 1|_{x>011} 0|_{x\le 011}$
$f(x \in \{0,1\}^3) = 1|_{x=\{010,011,110,111\}} 0|_{x=\{000,001,100,101\}}$
Interference
Entanglement
Superposition

Figure 43. Deutsch-Jozsa's QA: Step 1.4. Quantum gate assembly

Step 1.4 from Fig. 23 as QAG assembly for design cases is shown in Fig. 44.

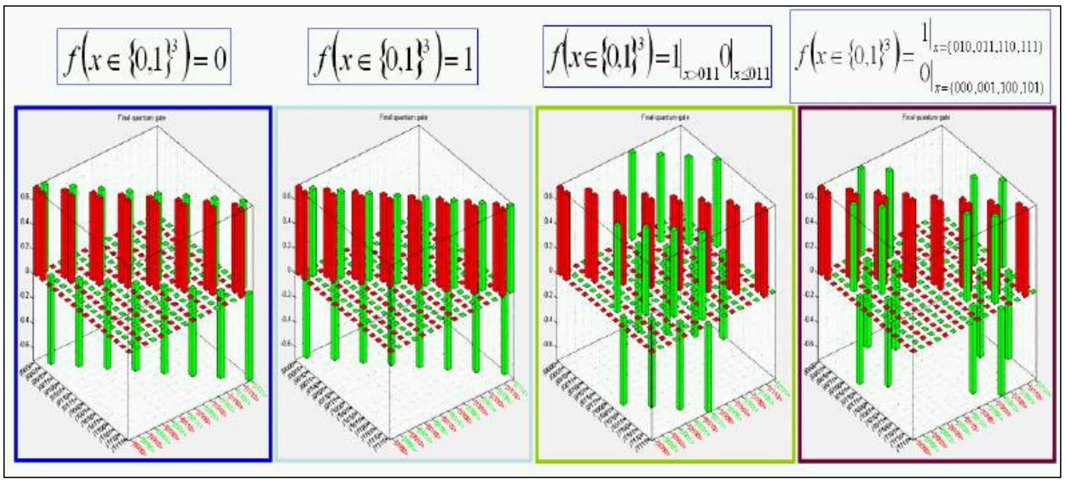
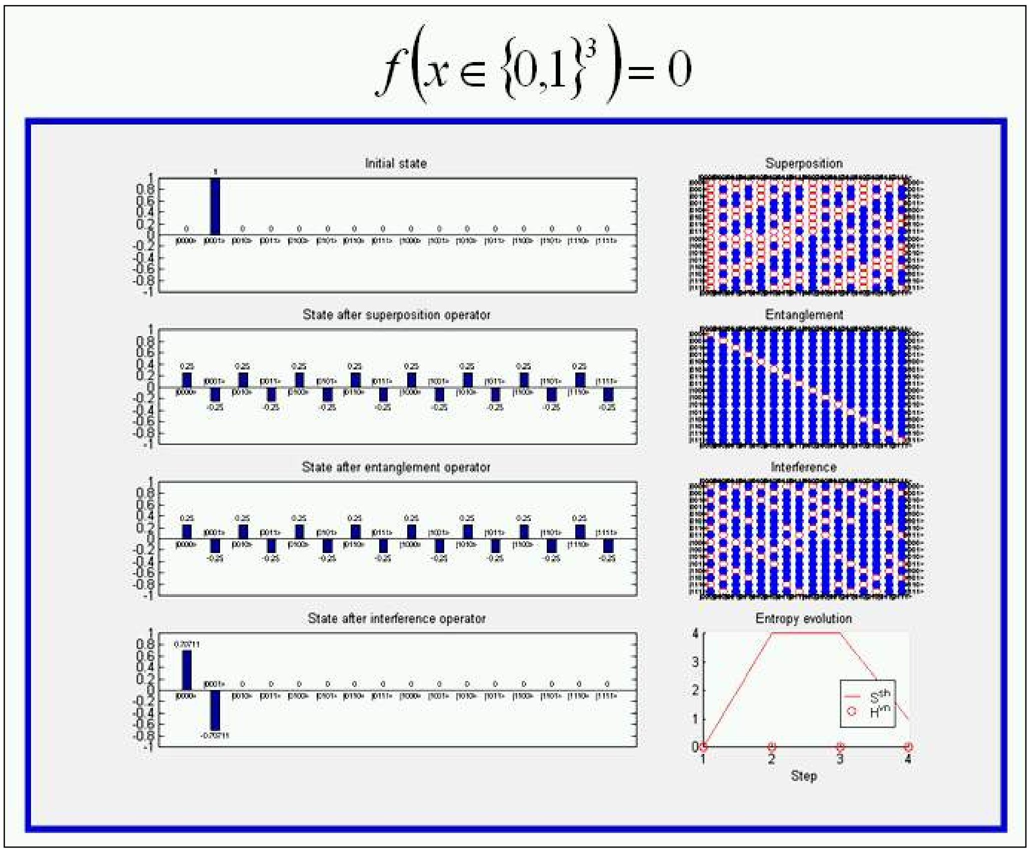


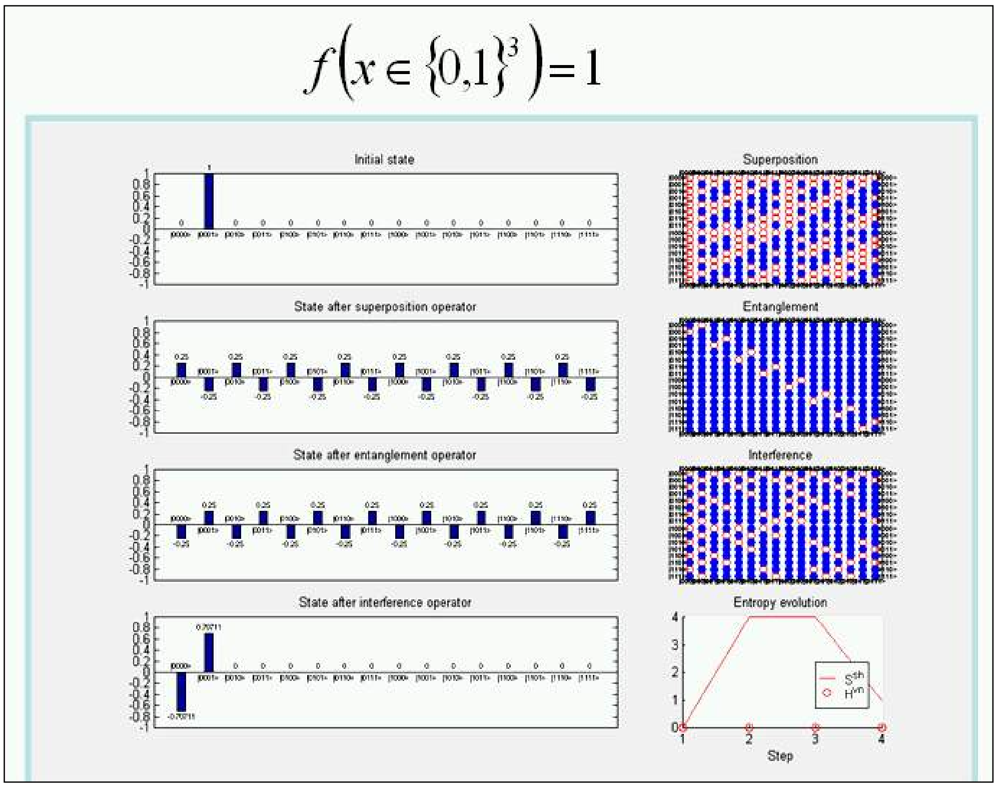
Figure 44. Deutsch-Jozsa's QA: Step 1.4. Assembled quantum gates

Figs 45 and 46 show the results of algorithm gate execution for constant and balanced functions respectively (corresponding to step 2 from Fig. 23).



$f\left(x \in \{0,1\}^3\right) = 0$
Initial state
State after superposition operator
State after entanglement operator
State after interference operator
Superposition
Entanglement
Interference
Entropy evolution
S^{sh}
H^{vn}
Step

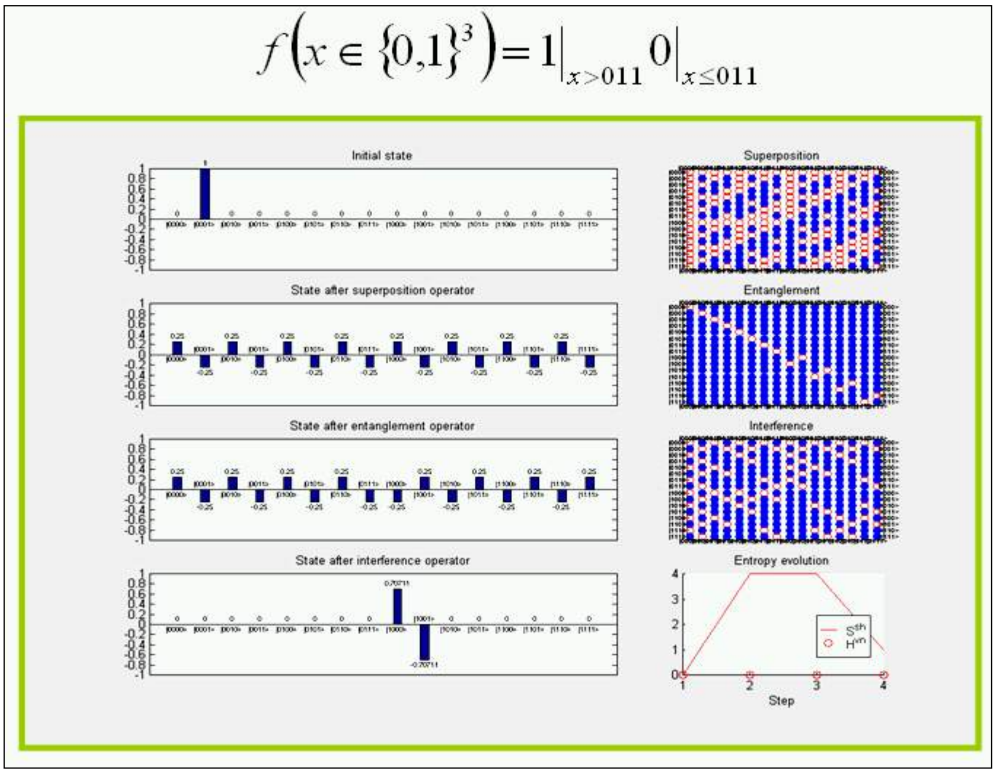
(a)



$f\left(x \in \{0,1\}^3\right) = 1$
Initial state
State after superposition operator
State after entanglement operator
State after interference operator
Superposition
Entanglement
Interference
Entropy evolution
S^{sh}
H^{vn}
Step

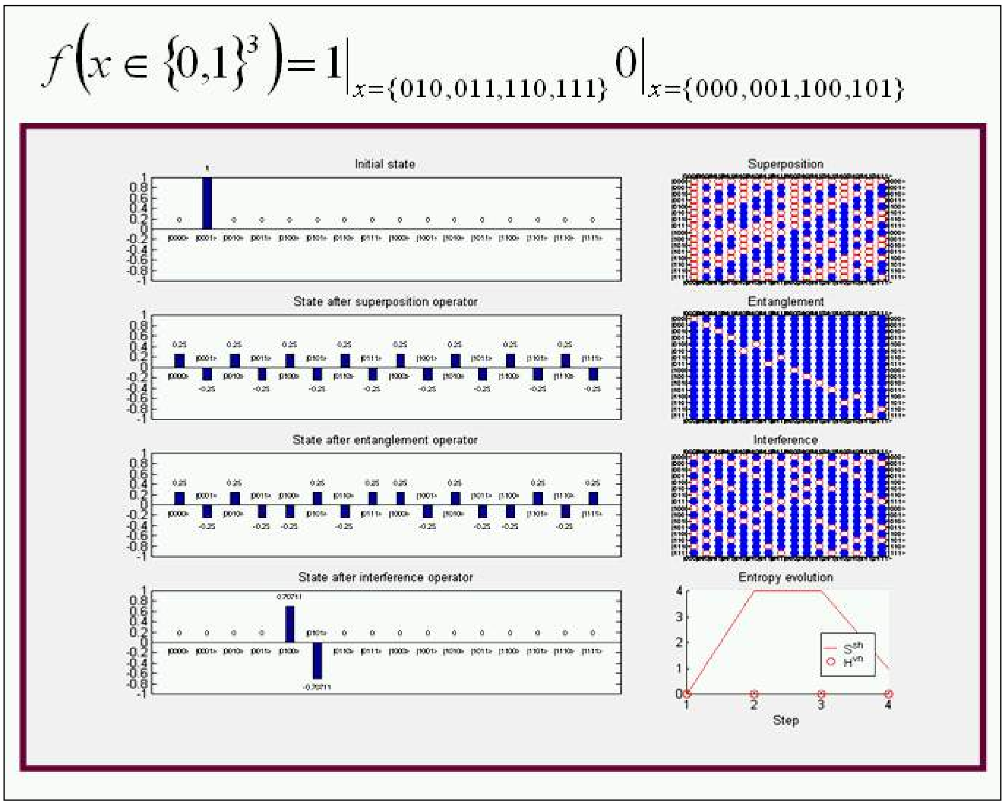
(b)

Figure 45. Deutsch-Jozsa's QA: Step 2. Algorithm execution: Constant functions



$$f\left(x \in \{0,1\}^3\right) = 1\big|_{x>011} 0\big|_{x\leq 011}$$

(a)



$$f\left(x \in \{0,1\}^3\right) = 1\big|_{x=\{010,011,110,111\}} 0\big|_{x=\{000,001,100,101\}}$$

(b)

Figure 46. Deutsch-Jozsa's QA: Step 2. Algorithm execution: Balanced functions

Fig. 47 shows the 3D result of amplitude probability evolution for Deutsch-Jozsa's QA execution.

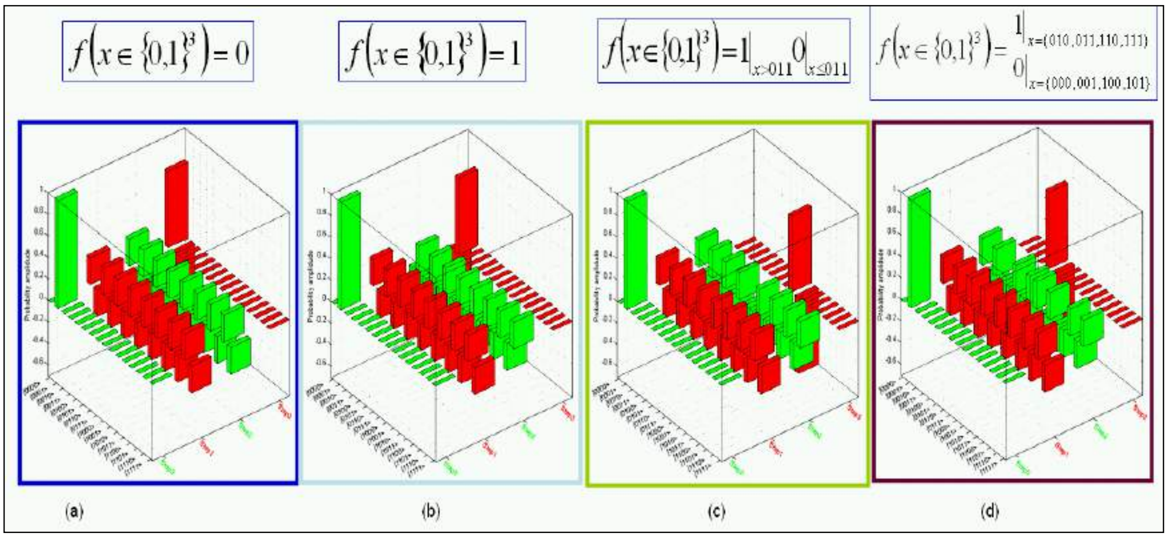


Figure 47. Deutsch-Jozsa's QA: Step 2. Algorithm execution 3d dynamics: Probability amplitudes

Fig. 48 shows the 3D result of probability evolution of this QA.

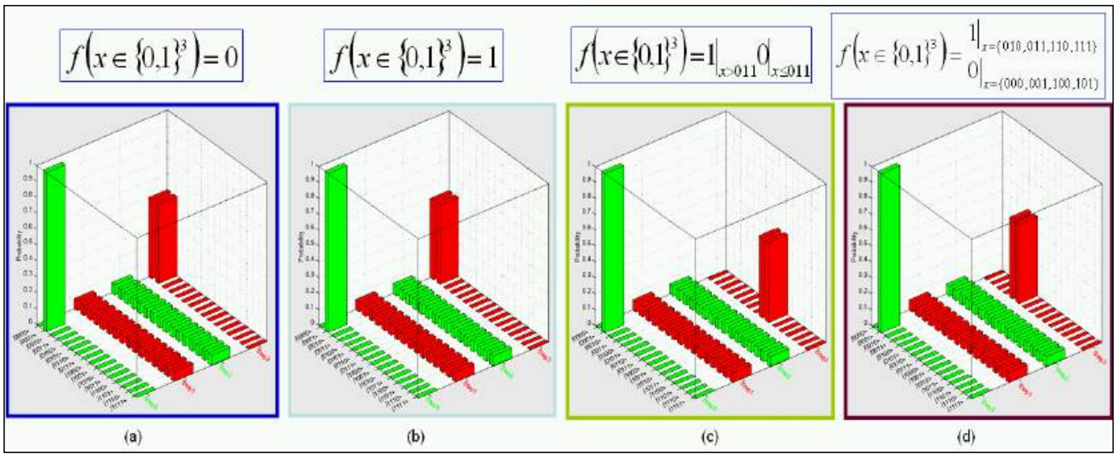


Fig. 48. Deutsch-Jozsa's QA: Step 2. Algorithm execution 3d dynamics: Probabilities

Result interpretation (corresponding to step 2.4 from Fig. 23) is shown in Fig. 49.

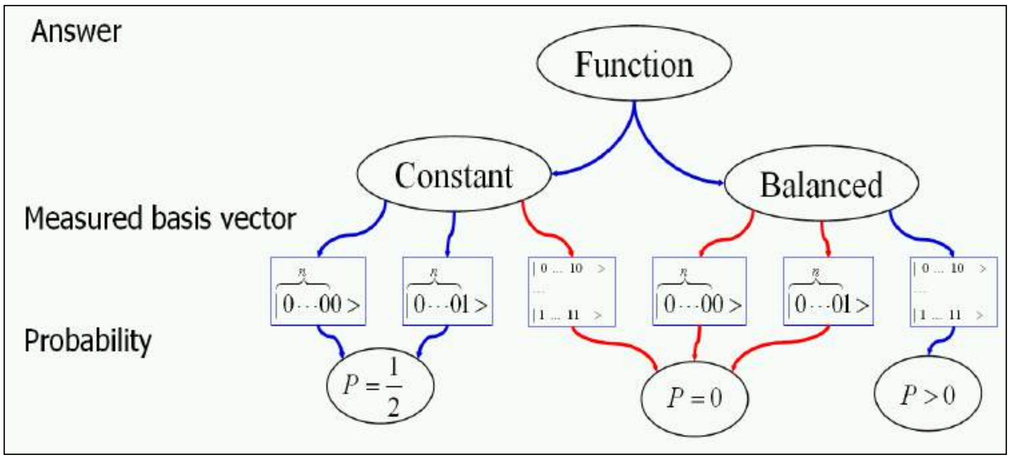


Figure 49. Deutsch-Jozsa's QA: Step 2.4 Result interpretation

In Deutsch-Jozsa's QA, the mathematical and physical structures of the interference operator $\left({}^{n}H \otimes I\right)$ differ from its superposition operator $\left({}^{n+1}H\right)$. The interference operator extracts the qualitative information about the property (constant or balanced property of function *f*) with operator ${}^{n}H$, and separate this property qualitatively with operator *I*. Deutsch-Jozsa's QA is a decision making algorithm.

For the case of Deutsch-Jozsa's QA only one iteration is needed without estimation quantitatively the qualitative property of function *f* and with error probability 0.5 of successful result. It means that the Deutsch-Jozsa QA is a robust QA.

The superposition operator organizes the quantum massive parallel computation process and robust extraction of function property is provided by the entanglement operator. The next section illustrates the Simon's QA and the role of the interference operator in the searching problem.

Simon's algorithm

Simon's algorithm is now illustrated using circuits and pointing out the role of interference.

Simon's problem. Simon's problem is formulated as follows:

Input	f:$\{0,1\}^{n} \to \{0,1\}^{n}$: $\exists s\in\{0,1\}^{n}-\{0..0\}:\forall x,y\in\{0,1\}^{n}$: $f(x) = f(y) \Leftrightarrow (x = y \vee x = y\oplus s)$
Problem	*Find s*

Encoder. As for the Deutsch-Jozsa's algorithm, firstly consider some special cases.

A. Introductory example. Consider the case:

$$n=2,\ f(00)=00,\ f(01)=01,\ s=11.$$

Then, the *f* map table is:

(x_0, x_1)	***f*(x_0, x_1)**
00	00
01	01
10	01
11	00

Step 1

Function *f* is encoded into the injective function *F* built in the usual way:

$$F:\{0,1\}^{n+n} \to \{0,1\}^{n+n} \quad \textit{such that}$$

$$F(x_0,x_1,..,x_{n-1},y_0,y_1,..,y_{n-1})=\left(x_0,x_1,..,x_{n-1},f(x_0,x_1,..,x_{n-1})\oplus(y_0,y_1,..,y_{n-1})\right)$$

This is the *F* map table:

(x_0,.., x_{n-1}, y_0,.., y_{n-1})	***F*(x_0,.., x_{n-1}, y_0,.., y_{n-1})**
0000	0000
0100	0101
1000	1001
1100	1100
0001	0001
0101	0100

$(x_0,.., x_{n-1}, y_0,.., y_{n-1})$	$F(x_0,.., x_{n-1}, y_0,.., y_{n-1})$
1001	1000
1101	1101
0010	0010
0110	0111
1010	1011
1110	1110
0011	0011
0111	0110
1011	1010
1111	1111

Step 2

Now encode F map table into U_F map table. As usually, the rule is:

$$\forall t \in \{0,1\}^{n+n}: U_F\,[\tau(t)] = \tau[F(t)],$$

where τ is the code map defined by Eq. (2) above. This means:

$\|x_0.. x_{n-1}\, y_0.. y_{n-1}>$	$U_F\|x_0.. x_{n-1}\, y_0.. y_{n-1}>$
|0010>	|0010>
|0110>	|0111>
|1010>	|1011>
|1110>	|1110>
|0011>	|0011>
|0111>	|0110>
|1011>	|1010>
|1111>	|1111>

Step 3

Using the rule:

$$[U_F]_{ij} = 1 \Leftrightarrow U_F \left| j \right\rangle = \left| i \right\rangle$$

calculate U_F as a block matrix:

U_F	|00>	|01>	|10>	|11>
|00>	I⊗I	0	0	0
|01>	0	I⊗C	0	0
|10>	0	0	I⊗C	0
|11>	0	0	0	I⊗I

This matrix preserves the first two vectors in the input tensor product vector. It preserves the last two when the first two vectors are |0> and |0> or |1> and |1>. It preserves the third vector, but it flips the fourth, when the first two vectors are |0> and |1> or |1> and |0>. Observe that the block matrix in cell (i, i) is identical to the block matrix in cell $(i\oplus s, i\oplus s)$, where i is the binary label of the vector marking the matrix row and column of the cell.

B. *General case with* n = 2. In general, if n = 2, repeating steps 1, 2 and 3 as for Deutsch-Jozsa's algorithm, the general operator U_F is obtained in the following form:

U_F	|00>	|01>	|10>	|11>
|00>	M_{00}	0	0	0
|01>	0	M_{01}	0	0
|10>	0	0	M_{10}	0
|11>	0	0	0	M_{11}

where $M_i \in \{I\otimes I, I\otimes C, C\otimes I, C\otimes C\}$ and $M_i = M_j \Leftrightarrow (j = i \vee j = i \oplus s)$.

C. *General case*. Generalizing the results obtained in the previous examples and reasoning like in Deutsch-Jozsa's algorithm, one can find the structure of U_F for Simon's algorithm too. The final matrix is:

U_F	|0..0>	|0..1>	…	|1..1>
|0…0>	$M_{0\ldots0}$	0	…	0
|0…1>	0	$M_{0\ldots1}$	…	0
…	…	…	…	...
|1...1>	0	0	0	$M_{1\ldots1}$

where $M_i = P_1 \otimes\ldots\otimes P_n$, $P_k \in \{I, C\}$, $k=1,\ldots,n$ and $M_i = M_j \Leftrightarrow (j = i \vee j = i \oplus s)$.

Note that the column labels are basis vectors of dimension n (not $2n$).

Quantum block In Fig. 50 (a) shows the circuit describing Simon's QG.

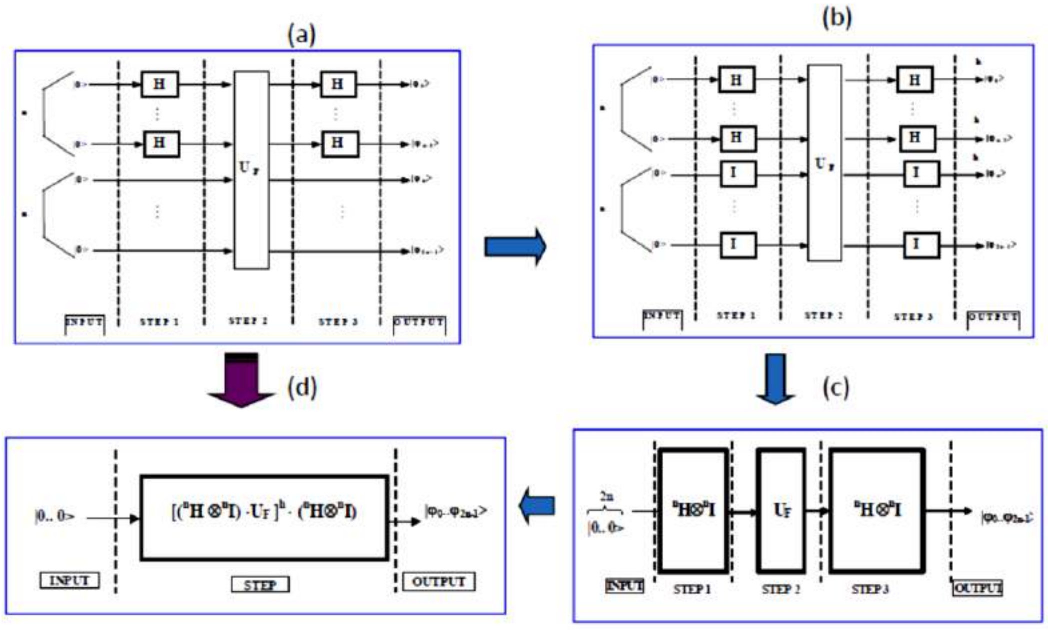


Figure 50. Simon's quantum algorithm simulation: Circuit representation and corresponding gate design

Using the transformation rules defined in Fig. 23 this circuit is complied into the corresponding gate.

Fig. 50 (d) shows Simon's QAG. To calculate this gate and establish what output vector it produces, it is first useful to deal with the introductory example of Section A, passing then to the general case with $n = 2$. Finally, the gate structure is described in the general situation ($n > 0$).

A. *Introductory example*. In the case considered before ($n = 2$, $f(00) = 00$, $f(01) = 01$, $s = 11$), the QAG assumes this form:

$$G=(^nH \otimes ^nI) \cdot U_F \cdot (^nH\otimes ^nI) \tag{3}$$

where U_F has been calculated in Section A, Step 2.

Start finding matrix ${}^2H\otimes{}^2I$, using the results about the tensor power of matrix H obtained in previous section.

${}^2H\otimes{}^2I$	|00>	|01>	|10>	|11>
|00>	${}^2I/2$	${}^2I/2$	${}^2I/2$	${}^2I/2$
|01>	${}^2I/2$	$-{}^2I/2$	${}^2I/2$	$-{}^2I/2$
|10>	${}^2I/2$	${}^2I/2$	$-{}^2I/2$	$-{}^2I/2$
|11>	${}^2I/2$	$-{}^2I/2$	$-{}^2I/2$	${}^2I/2$

Recall matrix U_F and calculate G:

U_F	|00>	|01>	|10>	|11>
|00>	^{2}I	0	0	0
|01>	0	I⊗C	0	0
|10>	0	0	I⊗C	0
|11>	0	0	0	^{2}I

$U_F\cdot({}^2H\otimes{}^2I)$	|00>	|01>	|10>	|11>
|00>	${}^2I/2$	${}^2I/2$	${}^2I/2$	${}^2I/2$
|01>	$I\otimes C/2$	$-I\otimes C/2$	$I\otimes C/2$	$-I\otimes C/2$
|10>	$I\otimes C/2$	$I\otimes C/2$	$-I\otimes C/2$	$-I\otimes C/2$
|11>	${}^2I/2$	$-{}^2I/2$	$-{}^2I/2$	${}^2I/2$

G	|00>	|01>	|10>	|11>
|00>	$({}^2I+I\otimes C)/2$	0	0	$({}^2I-I\otimes C)/2$
|01>	0	$({}^2I+I\otimes C)/2$	$({}^2I-I\otimes C)/2$	0
|10>	0	$({}^2I-I\otimes C)/2$	$({}^2I+I\otimes C)/2$	0
|11>	$({}^2I-I\otimes C)/2$	0	0	$({}^2I+I\otimes C)/2$

With G from (3.2) having this structure, apply it to vector |0000> to obtain the following result:

$$G|0000\rangle=|00\rangle\frac{\left({}^2I+I\otimes C\right)}{2}|00\rangle+|11\rangle\frac{\left({}^2I-I\otimes C\right)}{2}|00\rangle .$$

This means:

$$G|0000\rangle=\frac{1}{2}|00\rangle\left(|00\rangle+|01\rangle\right)+\frac{1}{2}|11\rangle\left(|00\rangle-|01\rangle\right).$$

If the output vector is measured, one can obtain only 4 possible results: |0000>, |0001>, |1100> and |1101>. Encode back into their binary labels the values of the first two basis vectors of dimension 2 in the output tensor product: these labels are 00 or 11. Then solve the system:

$$\begin{cases}(00)\cdot(t_1t_2)=0\\(11)\cdot(t_1t_2)=0\\t_1\neq 0,t_2\neq 0\end{cases}\Rightarrow\begin{cases}0\cdot t_1\oplus 0\cdot t_2=0\\1\cdot t_1\oplus 1\cdot t_2=0\\t_1\neq 0,t_2\neq 0\end{cases}\Rightarrow\begin{cases}0\oplus 0=0\\t_1\oplus t_2=0\\t_1\neq 0,t_2\neq 0\end{cases}\Rightarrow\begin{cases}t_1\oplus t_2=0\\t_1\neq 0,t_2\neq 0\end{cases}\Rightarrow\begin{cases}t_1=1\\t_2=1\end{cases}.$$

Since s=(11), then $s=(t_1, t_2)$. Therefore, s can be calculated as the solution of the system:

$$\begin{cases}(00)\cdot s=0\\(11)\cdot s=0\\s\neq(0,0)\end{cases}$$

B. *General case with n* = 2. In the general case with n=2, matrix U_F has the form:

U_F	|00>	|01>	|10>	|11>
|00>	M_{00}	0	0	0
|01>	0	M_{01}	0	0
|10>	0	0	M_{10}	0
|11>	0	0	0	M_{11}

where $M_i \in \{I\otimes I, I\otimes C, C\otimes I, C\otimes C\}$ and $M_i = M_j \Leftrightarrow (j=i \vee j=i\oplus s)$.

Using matrix ${}^2H\otimes{}^2I$ calculated above, obtain:

$U_F\cdot({}^2H\otimes{}^2I)$	|00>	|01>	|10>	|11>
|00>	$M_{00}/2$	$M_{00}/2$	$M_{00}/2$	$M_{00}/2$
|01>	$M_{01}/2$	$-M_{01}/2$	$M_{01}/2$	$-M_{01}/2$
|10>	$M_{10}/2$	$M_{10}/2$	$-M_{10}/2$	$- M_{10}/2$
|11>	$M_{11}/2$	$-M_{11}/2$	$-M_{11}/2$	$M_{11}/2$

G	|00>	|01>	|10>	|11>
|00>	$(M_{00}+M_{01}+M_{10}+M_{11})/4$	$(M_{00}-M_{01}+M_{10}-M_{11})/4$	$(M_{00}+M_{01}-M_{10}-M_{11})/4$	$(M_{00}-M_{01}-M_{10}+M_{11})/4$
|01>	$(M_{00}-M_{01}+M_{10}-M_{11})/4$	$(M_{00}+M_{01}+M_{10}+M_{11})/4$	$(M_{00}-M_{01}-M_{10}+M_{11})/4$	$(M_{00}+M_{01}-M_{10}-M_{11})/4$
|10>	$(M_{00}+M_{01}-M_{10}-M_{11})/4$	$(M_{00}-M_{01}-M_{10}+M_{11})/4$	$(M_{00}+M_{01}+M_{10}+M_{11})/4$	$(M_{00}-M_{01}+M_{10}-M_{11})/4$
|11>	$(M_{00}-M_{01}-M_{10}+M_{11})/4$	$(M_{00}+M_{01}-M_{10}-M_{11})/4$	$(M_{00}-M_{01}+M_{10}-M_{11})/4$	$(M_{00}+M_{01}+M_{10}+M_{11})/4$

Now, consider the following cases:

(i) s = 01; (ii) s = 10; (iii) s = 11.

In the first case $M_{00}=M_{01}\neq M_{10}=M_{11}$. This means:

G_{01}	|00>	|01>	|10>	|11>
|00>	$(M_{00}+M_{10})/2$	0	$(M_{00}-M_{10})/2$	0
|01>	0	$(M_{00}+M_{10})/2$	0	$(M_{00}-M_{10})/2$
|10>	$(M_{00}-M_{10})/2$	0	$(M_{00}+M_{10})/2$	0
|11>	0	$(M_{00}-M_{10})/2$	0	$(M_{00}+M_{10})/2$

In the second case $M_{00}=M_{10}\neq M_{01}=M_{11}$. This means:

G_{10}	|00>	|01>	|10>	|11>
|00>	$(M_{00}+M_{01})/2$	$(M_{00}-M_{01})/2$	0	0
|01>	$(M_{00}-M_{01})/2$	$(M_{00}+M_{01})/2$	0	0
|10>	0	0	$(M_{00}+M_{01})/2$	$(M_{00}-M_{01})/2$
|11>	0	0	$(M_{00}-M_{01})/2$	$(M_{00}+M_{01})/2$

Finally, in the third case $M_{00}=M_{11}\neq M_{01}=M_{10}$. This means:

G_{11}	|00>	|01>	|10>	|11>
|00>	$(M_{00}+M_{01})/2$	0	0	$(M_{00}-M_{01})/2$
|01>	0	$(M_{00}+M_{01})/2$	$(M_{00}-M_{01})/2$	0
|10>	0	$(M_{00}-M_{01})/2$	$(M_{00}+M_{01})/2$	0
|11>	$(M_{00}-M_{01})/2$	0	0	$(M_{00}+M_{01})/2$

Consider the application of G_{01}, G_{10} and G_{11} to vector |0000> in the three cases:

Case	s	Output vector: G_s|0000>
1	01	G_{01}|0000>=1/2 |00>($M_{00}+M_{10}$)|00> + 1/2 |10>($M_{00}-M_{10}$)|00>
2	10	G_{10}|0000>=1/2 |00>($M_{00}+M_{01}$)|00> + 1/2 |01>($M_{00}-M_{01}$)|00>
3	11	G_{11}|0000>=1/2 |00>($M_{00}+M_{01}$)|00> + 1/2 |11>($M_{00}-M_{01}$)|00>

Measure the output vector in these three cases and encode back into binary values the first two basis vectors in the tensor product, to obtain the following result:

Case	s	Binary Values *(FROM THE FIRST TWO VECTORS)*	Probabilities
1	01	$(a, b)=(0,0)$	0.5
		$(a, b)=(1,0)$	0.5
2	10	$(a, b)=(0,0)$	0.5
		$(a, b)=(0,1)$	0.5
3	11	$(a, b)=(0,1)$	0.5
		$(a, b)=(1,1)$	0.5

Note that $(a, b) \cdot s = 0$ where a and b are the binary values from the first two vectors. The equations so generated let us find s as the solution of the corresponding system.

C. *General case* ($n > 0$). Now consider a general positive value for number n.

The operator U_F is:

U_F	$\|0..0>$	$\|0..1>$	…	$\|1..1>$
$\|0..0>$	$M_{0..0}$	0	…	0
$\|0..1>$	0	$M_{0..1}$	…	0
…	…	…	…	…
$\|1..1>$	0	0	0	$M_{1..1}$

where $M_i = P_1 \otimes .. \otimes P_n$, $P_k \in \{I, C\}$, $k = 1,..,n$ and and $M_i = M_j \Leftrightarrow (j=i \vee j=i\oplus s)$.

Operator ${}^nH\otimes {}^nI$ is built from operator ${}^nH$:

${}^nH\otimes{}^nI$	$\|0..0>$	$\|0..1>$	…	$\|j>$	…	$\|1..1>$
$\|0..0>$	${}^nI/2^{n/2}$	${}^nI/2^{n/2}$	…	${}^nI/2^{n/2}$	…	${}^nI/2^{n/2}$
$\|0..1>$	${}^nI/2^{n/2}$	$-{}^nI/2^{n/2}$	…	$(-1)^{(0..1)\cdot j}\,({}^nI/2^{n/2})$	…	$-{}^nI/2^{n/2}$
…	…	…	…	…	…	…
$\|i>$	${}^nI/2^{n/2}$	$(-1)^{i\cdot(0..1)}$ $({}^nI/2^{n/2})$	…	$(-1)^{i\cdot j}\,({}^nI/2^{n/2})$	…	$(-1)^{i\cdot(1..1)}\,({}^nI/2^{n/2})$
…	…	…	…	…	…	…
$\|1..1>$	${}^nI/2^{n/2}$	$-{}^nI/2^{n/2}$	…	$(-1)^{(1..1)\cdot j}\,({}^nI/2^{n/2})$	…	$(-1)^{(1..1)\cdot(1..1)}\,({}^nI/2^{n/2})$

$U_F\cdot({}^nH\otimes{}^nI)$	$\|0..0>$	…	$\|j>$	…	$\|1..1>$
$\|0..0>$	$M_{0..0}/2^{n/2}$	…	$M_{0..0}/2^{n/2}$	…	$M_{0..0}/2^{n/2}$
…	…	…	…	…	…
$\|i>$	$M_i/2^{n/2}$	…	$(-1)^{i\cdot j}\,M_i/2^{n/2}$	…	$(-1)^{i\cdot(1..1)}\,M_i/2^{n/2}$
…	…	…	…	…	…
$\|1..1>$	$M_{1..1}/2^{n/2}$	…	$(-1)^{(1..1)\cdot j}\,M_{1..1}/2^{n/2}$	…	$(-1)^{(1..1)\cdot(1..1)}$ $M_{1..1}/2^{n/2}$

The first column of the final gate has the following form:

G	$\|0..0>$	…
$\|0..0>$	$(M_{0..0}+..+M_i+..+M_{1..1})/2^n$	…
…	…	…
$\|i>$	$(\,\Sigma_{j\in\{0,1\}^n}(-1)^{i\cdot j}M_j\,)/2^n$	…
…	…	…
$\|1..1>$	$(\,\Sigma_{j\in\{0,1\}^n}(-1)^{(1..1)\cdot j}M_j\,)/2^n$	…

The *interference* operator (${}^nH\otimes{}^nI$) creates the following term: $\frac{1}{2^n}\sum_{j\in\{0,1\}^n}(-1)^{i\cdot j}M_j$.

Since $M_h = M_k \Leftrightarrow (h = k \vee h = k \oplus s)$, then this term may be written as:

$$\frac{1}{2^n}\sum_{j\in\{0,1\}^n}(-1)^{i\cdot j}M_j = \frac{1}{2^n}\sum_{k\in S}\left[(-1)^{i\cdot k}+(-1)^{i\cdot(k\oplus s)}\right]M_k = \frac{1}{2^n}\sum_{k\in S}(-1)^{i\cdot k}\left[1+(-1)^{i\oplus s}\right]M_k \ ,$$

where S is such that:

$$\neg\exists x,y\in S : x\oplus s = y \quad \neg\exists x,y\in\{0,1\}^n - S : x\oplus s = y.$$

The gate can be rewritten in this way:

G	|0..0>	…
|0…0>	$\Sigma_{k\in S}\,(-1)^{(0..0)\cdot k}[1+(-1)^{(0\ldots0)\cdot s}]M_k/2^n$	…
…	…	…
|i>	$\Sigma_{k\in S}\,(-1)^{i\cdot k}[1+(-1)^{i\cdot s}]M_k/2^n$	…
…	…	…
|1…1>	$\Sigma_{k\in S}\,(-1)^{(1..1)\cdot k}[1+(-1)^{(1\ldots1)\cdot s}]M_k/2^n$	…

The term $[1+(-1)^{i\cdot s}]$ is 0 if and only if $i \cdot s = 1$. So, only those cells in the column that are labeled by $|i>$ such that $i \cdot s = 0$ are non-null. This means that:

$$G|0..00..0\rangle = \frac{1}{2^{n-1}}\sum_{i\in\{0,1\}^n : i\cdot s=0}|i\rangle.$$

The quantum block ends with measurement, which therefore produces a basis vector $|i>$ such that $i \cdot s = 0$. Thus, the interference operator $\left({}^nH \otimes {}^nI\right)$ is created the important component of final result and with the entanglement operator and a measurement process can extract this final result.

In this case, the interference operator $\left({}^nH \otimes {}^nI\right)$ extracts the qualitative property of the function f using operator $\left({}^nH\right)$ and estimates quantitatively this property as a solution number with operator $\left({}^nI\right)$.

With tensor product $(\otimes)$ the interference operator $\left({}^nH \otimes {}^nI\right)$ joins both possibilities in one automation operation.

Simon's QA is the *search* algorithm and this property is described by specific structure of interference operator. Operator $\left({}^nH\right)$ from interference operator $\left({}^nH \otimes I\right)$ in Deutsch-Jozsa's QA created the distribution of probability amplitude with the same amplitude of probability $|\alpha_0| = |\alpha_1| = \frac{1}{\sqrt{2}}$; the measurement with the ancillae qubit as $\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\otimes I$ extracts only the qualitative information about the function f, and gives sufficient and necessary values of probability 0.5 for the separation of solutions in decision-making algorithm. The quantum block for this QA is repeated only one time and the final collection is made only one basic vector.

For state vector $|i\rangle$ of Deutsch-Jozsa's QA the estimator of amplitude of probability is:

$$\left(\frac{1}{2^n}\sum_{j\in\{0,1\}^n}(-1)^{i\cdot j}M_j\right)H \ .$$

The operator H in this operator plays the role of the destructive interference for ancillary qubit: $H\left(\frac{1}{\sqrt{2}}\left[|0\rangle-|1\rangle\right]\right)=|0\rangle$ and realize toss and coin procedure of random measurement. The different signs $\left(\pm\frac{1}{\sqrt{2}}\right)$ of amplitude probability in ancillary qubit with identity operator I guarantee the recognition the solution with toss and coin procedure of measurement that is necessary and sufficient conditions for successful result of quantum computing.

In the case of Simon's QA the estimator of amplitude of probability in state vector $|i\rangle$ is as follows:

$$\frac{1}{2^n}\sum_{k\in S}(-1)^{i\cdot j}M_j=\frac{1}{2^n}\sum_{k\in S}(-1)^{i\cdot k}\left[\overset{\substack{\text{Choice of}\\\text{solution}}}{\boxed{1}}+\underbrace{(-1)^{1\oplus s}}_{\substack{\text{Quantitave solution}\\\text{estimator}}}\right]M_k\ ,$$

where S is such that include almost quantitative information about the solutions.

The estimator of amplitude probability in Simon's QA constructively distributed the amplitude of probability: increase the amplitude for «good» solution with quantitative estimation of this solution and decrease other amplitudes of solution probability.

In Simon's QA the role of interference is different than in Deutsch-Jozsa's QA while this operator extract the qualitative information about the solution and estimate quantitatively this solutions using n times iterations in quantum block.

The quantum block is repeated enough times to get enough information to determine s. Since every vector will constitute a coefficient vector for an equation where s is the variable vector, this number depends on how many different equations are needed in order to find s. Since s has length n, in general one will need a number n of different equations. This requires, in general, a linear number of measurements.

Decoder. The quantum block is repeated $O(n)$ times until a collection of n different vectors have been generated. As for the case $n = 2$, for every vector in this collection, the first n basis vectors of dimension 2 composing it through tensor product are encoded back into their binary values. In this way they can be used as coefficients for building an equation whose variables are the bits of s. By solving the system made of these equations, one can find s.

Simon's QA is the benchmark of the search QA family and separates this family from the decision making QA family using the special description form of interference operator. This algorithm has a mathematical structure similar to the superposition operator $\left({}^nH\otimes{}^nI\right)$ but has different physical meaning.

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